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On quantifier-rank equivalence between linear orders

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ABSTRACT

We construct winning strategies for both players in the Ehrenfeucht–Fraïssé game on linear orders. To this end, we define the local quantifier-rank k theory of a linear order with a single constant $Th_k^{loc}(\lambda, x)$, and prove a normal form for $\equiv_k$ classes, expressed in terms of local classes. We describe two implications of this theorem: 1. a decision procedure for whether a set U of pairs of $\equiv_k$ classes is consistent – whether for some linear order λ, U is the set of pairs (ϕ, ψ) such that $\lambda \models \exists x(\phi^{<x} \wedge \psi^{>x})$ – which runs in time linear in the size of the formula which expresses that exactly the pairs of $\equiv_k$ classes in U are realized. The only obstacle to effectively listing the theory of linear order is the vast number of different $\equiv_k$ classes of theories of linear order. 2. We find a finitely axiomatizable linear order λ which we construct inside any $\equiv_k$ class of linear orders. We relate our winning strategies to semimodels of the theory of linear order. First, we situate our result in a historical background.

0. Introduction

That the theory of ordinals is decidable is proved in [2,10,11] as an initial step in other directions. In [4] we find the decidability of the theory of ordinals proved from a game-theoretic view. The reader of this paper must be aware of the game defined in [4]. We will not use any theorem from [4], but that paper is, in any case, the right introduction to our subject. In [6] (Theorem 3, p. 411) we find infinitarily equivalent ordinals, generalizing the $\equiv_k$-equivalent ordinals of [4] (Theorem 12). On the other hand, Scott sentences of infinitary logic define any ordinal. In [8] we find smaller $\equiv_k$ ordinals than those in [6] as a result of analyzing the Ehrenfeucht–Fraïssé game (hereafter, the *EF* game) to the precise solution of Theorem 0.1:

For any ordinals μ, δ, let δ be a cutoff, with respect to which we see μ as having a *body* and *tail*: $\beta_\delta(\mu) = \{x \in \mu : x + \omega^\delta \leq \mu\}$, $\tau_\delta(\mu) = \{x \in \mu : x + \omega^\delta > \mu\}$. The separation of μ into these two pieces is useful in defining the Cantor Normal Form (hereafter CNF). Note that $\mu = \beta_\delta(\mu) + \tau_\delta(\mu)$ and ω^δ divides $\beta_\delta(\mu)$.

Theorem 0.1 [8]. *For any ordinals μ_0, μ_1, α, $\mu_0 \not\equiv_\alpha \mu_1$ holds just in case for some $\delta < \alpha$, one of the following holds*:

1. $2 \times \delta < \alpha$ *and* $((\delta > 0) \wedge (\omega^\delta < \mu_0 \iff \omega^\delta < \mu_1)) \vee ((\delta = 0) \wedge (0 < \mu_0 \iff 0 < \mu_1))$,
2. $(\exists \gamma (2 \times \delta < \gamma < \alpha))$ *and*

$$\vee_{i<2}((\beta_{\delta+1}(\mu_i) = \beta_\delta(\mu_i)) \wedge (\beta_{\delta+1}(\mu_{1-i}) \neq \beta_\delta(\mu_{1-i}))\wedge$$

$$((\tau_\delta(\mu_i) = \emptyset) \vee (\tau_\delta(\mu_i) \setminus \{\beta_\delta(\mu_i)\}) \not\equiv_{\alpha-1} \omega^{\delta+1} + \tau_\delta(\mu_{1-i}))),$$

3. $\chi_\delta(\mu_0) \neq \chi_\delta(\mu_1)$ *and* $\vee_{i<2}\chi_\delta(\mu_i) < 2^{\alpha-(2\times\delta)} - 1$,

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where $\chi_\delta(\mu_i)$ *is*:

- *the number of* x *such that* $\omega^\delta \times x < \mu_i$ *and* $\omega^\delta \times x + \omega^{\delta+1} > \mu_i$,
- $+3$ *if* $\omega^{\delta+1} \le \mu_i$,
- -1 *if* $\delta > 0$ *and* $\mu_i < \omega^{\delta+1}$,
- -1 *if* $\tau_\delta(\mu_i) \neq \emptyset$ *and* $(\tau_\delta(\mu_i) \setminus \{\beta_\delta(\mu_i)\}) \not\equiv_{2\times\delta} \omega^\delta + \tau_\delta(\mu_i)$.

The Cantor Normal Form (hereafter CNF) of μ is similar to $\sum_\delta \omega^\delta \chi_\delta$. The first two conditions require μ_i to have the same CNF exponents δ; the final condition requires that the CNF coefficients are the same, or large. From the CNF of μ_0 and μ_1 we find a finite set of δ such that if the theorem fails at those δ, it must fail at all other δ. Theorem 0.1 describes the theory of ordinals precisely and can be used to imply other results. Our main theorem is, similarly, a solution to the *EF* game for linear orders:

Theorem 0.2. *If* λ *is a linear order and* k *is a finite number then there is a finite function* A_λ *mapping labels into* $\lambda \cup \lambda^+$ *such that for any two linear orders* λ *and* λ_0 *and any clock* k, $\lambda \equiv_k \lambda_0$ *holds just in case* A_λ *and* A_{λ_0} *have the same labels in their domains and induce the same ordering on them, and the sequences* $(Th^{\mathrm{loc}}_{k-1}(\lambda, a) : a \in A \cap \lambda)$ *and* $(\{Th^{\mathrm{loc}}_{k-1}(\lambda, a) : b < a < c\} : (b, c) \in A^+)$ *are identical*.

This theorem generalizes to $L_{\omega_1\omega}$, except that A is no longer finite; A is now an infinite tree of labeled elements with ranks B for each descending sequence B of ordinals in the quantifier rank ordinal α. It holds, too, for linear orders with unary relations. If we add unary relations to our vocabulary, there is an $\equiv^{\mathrm{loc}}_0$ class for each unary relation. With no unary relations, there is a unique $\equiv_0$ class, i.e., $(\lambda, x) \equiv_0 (\lambda, y)$ for all $x \in \lambda$ and $y \in \lambda$. Patterns and words in this vocabulary are described by almost locally closed sets, in Definition 4.2. Of course, these patterns arise even without unary predicates in the vocabulary. For instance, in the set of linear orders of the form $\lambda = \sum_{i\in\omega+Z\times\eta+\omega^*}(\eta + f(i)) + \eta$ for various functions f with domain $\omega + Z \times \eta + \omega^*$ and range n, a finite number, $\equiv^{\mathrm{loc}}_4$ classes define $\{(\lambda, x) : \exists i((i \in \omega + Z \times \eta + \omega^*) \wedge (x \in f(i)) \wedge ((f(i-1), f(i), f(i+1)) = (p, q, r))\}$ for each triple (p, q, r) of numbers ≤ 9. Thus, if we choose $n = 10$, then a consistent set of $\equiv^{\mathrm{loc}}_4$ sets is a set of triples (p, q, r) which can be strung together in a consistent way. There exist consistent sets of $\equiv^{\mathrm{loc}}_4$ classes for which the smallest minimal almost locally closed sets are very long.

This theorem continues the following line of research: The decidability of certain linear orders was studied in [12]. In [3] we find semimodels and the theorem that it is not easier to decide the theory of a semimodel and to relate that theory to that of a (infinite, normal, non-semi) model containing it, than to decide the theory of linear order. The decidability of the theory of linear order was proved in [7] using Ramsey's theorem and noting the monotonicity of $\{Th_k(\{z \in \lambda : x < z < y\}) : a < x < y < b\}$ as a function of a and b. Our proof of the same theorem avoids Ramsey's theorem. When we state this theorem in its local form, it can be used to prove many sentences of $L_{\omega_1\omega}$ to be consistent.

Theorem 0.3. *A set* U *of pairs of* $\equiv_k$ *classes is* $\{(Th_k(\{a \in \lambda : a < b\}), Th_k(\{a \in \lambda : a > b\}))\}$ *for some linear order* λ *just in case 1. there is an* $\equiv_k$ *class* $\xi(U)$ *such that for any* $(\phi, \psi) \in U$, *we have* $\exists x(\phi^{<x} \wedge \psi^{>x}) \equiv_k \xi(U)$, *and 2. there is a set* W *containing* U *and other sets of pairs of* $\equiv_k$ *classes such that every* $V \in W$ *has* $\xi(V)$ *as in part 1 and such that for any* $V \in W$ *and any element* $(\phi, \psi) \in V$ *there exist two elements* V_0, V_1 *of* W *such that* $(\xi(V_0), \xi(V_1)) = (\phi, \psi)$ *and* $V_0 + \{(\emptyset, \emptyset)\} + V_1 = V$.

In [1] we find a modification of the construction in [7] to generate a family of linear orders which intersects every $\equiv_k$ class and which all have not only decidable, but finitely axiomatizable theories. Theorem 0.3 permits the elimination of Ramsey's theorem also from our construction of a finitely axiomatizable model in any $\equiv_k$ class. A more practical sort of effectiveness can also be obtained: In [9] we learn that the theory of linear order is (decidable, but) intractable. In [5] we find an enumeration of the $\equiv_3$ classes of linear orders and the statement that current methods of deciding the theory of linear order cannot enumerate the $\equiv_4$ classes of linear orders. As a proof of concept we wrote a simple computer code which enumerates the 82988077686330 $\equiv_4$ classes of linear orders. On my laptop, it took the program two minutes to write those $8E13$ sentences. There are $4E23769$ sets of pairs of $\equiv_3$ classes, almost all of which are inconsistent. The computer avoids these by eliminating small inconsistent sentences, and then ignoring any theory which contains that sentence. We cannot list the $\equiv_5$ classes of linear order, but we can describe them both locally and globally. Size, not complexity, prevents the enumeration of $\equiv_5$. From the point of view of finite model theory, the most interesting results are the implications for semimodels. Semimodels are finite strings which often have interesting relationships to infinite models. Semimodels admit addition and multiplication, they can code the formation of a model by iterated application of functions (e.g., Skolem functions), and they handle local information quite easily.

1. Decidability: the consistency game

If ϕ and ψ are $\equiv_k$ classes of linear orders, then $\{\lambda + \mu : \lambda \in \phi, \mu \in \psi\}$ is contained in a single $\equiv_k$ class, which we call the sum of ϕ and ψ. If ϕ is an $\equiv_k$ class of linear orders, choose a sentence ϕ_0 in which the variable x does not appear which defines ϕ (that is, each linear order λ models ϕ_0 just in case it is in ϕ), and let $\phi^{<x}$ be the formula, with the variable x free, obtained from ϕ_0 by replacing every subformula $\exists y(\psi)$ by $\exists y((y < x) \wedge \psi)$ and replacing every subformula $\forall y(\psi)$ by $\forall y((y < x) \to \psi)$.

We define $\phi^{>x}$ similarly. We define $\phi^{(x,y)}$ similarly – by choosing a formula ϕ_0 which defines the $\equiv_k$ class ϕ and in which neither x nor y appears, and in ϕ_0 we replace every subformula $\exists z(\psi)$ by $\exists z((x < z < y) \wedge \psi)$ and replace every subformula $\forall z(\psi)$ by $\forall z((x < z < y) \rightarrow \psi)$. Because $\lambda \models_{[a/x]} \phi^{<x}$ just in case $\{m \in \lambda : m < a\} \models \phi_0$, which occurs just in case $\{m \in \lambda : m < a\} \in \phi$, we can read $\phi^{<x}$ as "ϕ holds left of x."

For any set U of pairs of $\equiv_k$ classes, let U_0 be the set of formulas $\{\exists x(\phi^{<x} \wedge \psi^{>x}) : (\phi, \psi) \in U\}$ and let $U_1 = \{\neg\exists x(\phi^{<x} \wedge \psi^{>x}) : (\phi, \psi)$ is a pair of $\equiv_k$ classes not in $U\}$, and let $\sigma_U = \wedge\{U_0, U_1\}$. For any $\equiv_{k+1}$ class μ with linear order $\lambda \in \mu$, the set $U(\lambda) = \{$pairs of $\equiv_k$ classes $(\phi, \psi) : \lambda \models \exists x(\phi^{<x} \wedge \psi^{>x})\}$ is such that $\sigma_{U(\lambda)}$ defines μ. Now $U(\lambda)$ depends only on the $\equiv_{k+1}$ class of μ (more, $\lambda_0 \equiv_{k+1} \lambda_1$ holds just in case $U(\lambda_0) = U(\lambda_1)$) so we write $U(\mu)$. The function from μ to $U(\mu)$ and from U to σ_U are inverses, i.e., μ is defined by $\sigma_{U(\mu)}$ and $U(\sigma_V) = V$ for any consistent set V of pairs of $\equiv_k$ classes.

If W is a set of sets U of pairs of $\equiv_k$ types for which there exists an $\equiv_k$ class $\xi(U)$ such that for any $(\phi, \psi) \in U, \exists x(\phi^{<x} \wedge \psi^{>x}) \equiv_k \xi(U)$, then we define addition on W:

$$U + V = \{(\phi, \psi + Th_k(1) + \xi(V)) : (\phi, \psi) \in U\} \cup \{(\xi(U) + Th_k(1) + \phi, \psi) : (\phi, \psi) \in V\}.$$

If σ_U and σ_V are consistent sentences, and hence elements of $\equiv_{k+1}$, then we define $\sigma_U + \sigma_V$ as above to be the $\equiv_{k+1}$ class of $\lambda + \mu$ for any/all λ such that $\lambda \models \sigma_U$ and any/all μ such that $\mu \models \sigma_V$. The notion of addition we have now defined for sets of pairs of $\equiv_k$ classes then agrees with addition on $\equiv_{k+1}$ classes: $\sigma_{U+V} = \sigma_U + \sigma_V$. So addition on W extends addition of $\equiv_{k+1}$ classes to some inconsistent sentences σ_U. This is important because we will search for the consistent sentences in W, using W's structure as a monoid.

Definition 1.1. The consistency game is as follows: On each turn, the game state is a finite sequence $(c_i : i < n)$ of constants and a sequence $(U_i : i \leq n)$ of sets of pairs of $\equiv_k$ classes each of which has an $\equiv_k$ class $\xi(U_i)$ such that for any $(\phi, \psi) \in U_i$, $\exists x(\phi^{<x} \wedge \psi^{>x}) \equiv_k \xi(U_i)$. The first player chooses $i \leq n$ and an element of U_i. The second player then adds a new constant c left of c_i and right of c_{i-1} and chooses two sets V_0 and V_1 of pairs of $\equiv_k$ classes and fixes for each $i < 2$ an $\equiv_k$ class $\xi(V_i)$ such that for any $(\phi, \psi) \in V_i, \exists x(\phi^{<x} \wedge \psi^{>x}) \equiv_k \xi(V_i)$. Player II loses unless the following conditions hold, in which case we say that player II has survived this round:

- $(\xi(V_0), \xi(V_1))$ is the element of U_i which player I chose, and
- $V_0 + \{(\emptyset, \emptyset)\} + V_1 = U_i$.

If player II has survived, then the game continues, with its game state $(c'_j : j < n+1)$ and $(U'_j : j \leq n+1)$ where $c'_j = c_j$ if $j < i$, $c'_{j+1} = c_j$ if $j \geq i$, and $c'_i = c$, the new constant; $U'_j = U_j$ if $j < i$, $U'_{j+1} = U_j$ if $j > i$, and $U'_i = V_0$, $U'_{i+1} = V_1$, where U_i has been replaced by V_0 and V_1. The initial state has $n = 0$, an empty sequence of constants, and a single set U_0 of pairs of $\equiv_k$ classes.

Now we prove Theorem 0.3: If W exists as in the statement of the theorem, then player II can play the consistency game indefinitely. If player II can play the consistency game indefinitely, and if player I exhausts every set U of pairs of $\equiv_k$ classes which is ever created during the game, then the set of constants played, with the ordering on each pair c_a, c_b determined at the moment when the latter was added to the set of constants, is a linear order C; we will prove that $C \models \sigma_{U_0}$. How can player I "exhaust" the set U if, when player I plays the first element of U, that set is immediately replaced by a pair of sets, V_0, V_1? Each element of the sum $V_0 + \{(\emptyset, \emptyset)\} + V_1$ corresponds to an element of U. In particular, $\{(\xi(V_0) + \emptyset, \emptyset + \xi(V_1))\}$ corresponds to the element that player I chose, after which U was replaced by V_0 and V_1. We say that player I exhausts U if player I plays elements in U, or in a summand such as V_0 or V_1 in a sequence of sets of pairs of $\equiv_k$ classes which sums to U, so that element of the summand corresponds to the desired element of U. Consider an interval T in C – either the interval left of c, right of c, or between c_a and c_b. Suppose that the parameters defining the interval existed already on the nth move, as c_i, for $i < n$, or as $c_{i_0} < c_{i_1}$, for $i_0 < i_1 < n$. Define U_T to be the sum of U_i over the interval – $(\sum_{j<i} U_j + \{(\emptyset, \emptyset)\}) + U_i$ for the interval left of c_i, $(\sum_{j>i, j<n} U_j + \{(\emptyset, \emptyset)\}) + U_n$ for the interval right of c_i, or $(\sum_{j>i_0, j<i_1} U_j + \{(\emptyset, \emptyset)\}) + U_{i_1}$ for the interval between c_{i_0} and c_{i_1}. Now each constant in the interval T in C corresponds to an element of U_T, since either the constant already existed in $(c_i : i < n)$, in which case there is a summand in U_T for it, or the constant was created on the nth move or later. In that case, when the constant is played, some U will be split into V_0 and V_1; each element of that U corresponds to some elements of U_T. On the other hand, if player I exhausts each set of pairs of $\equiv_k$ classes which is created, then U_T will be exhausted when each of its summands is exhausted. At that moment, there will be total functions mapping the interval T in C into the set U_T, and the set U_T into C, so that for $c \in T$ and $(\phi, \psi) \in U_T$, $(c, (\phi, \psi))$ being in either function or its inverse implies that the set U_T can be split into intervals V_0 and V_1 in W so that $\xi(V_0) = \phi$ and $\xi(V_1) = \psi$ and $V_0 + Th_k(1) + V_1 = U_T$. Now we prove the following: for each $j < k$, for each interval T in C, $T \equiv_j \xi(U_T)$. We prove this simultaneously for all intervals T, by induction on j – the correspondence between elements of U_T and elements of T such that (ϕ, ψ) corresponding to $c \in T$ implies that the interval of T left of c satisfies $Th_j(\phi)$ and the interval of T right of c satisfies $Th_j(\psi)$ is enough to show that T satisfies $Th_{j+1}(\xi(U_T))$. The base case, $j = 0$ is in fact the same argument: if U_T is empty, let n be the turn immediately after the last parameter was defined. Then either T is the interval left of c_0, or right of c_n, or between c_i and c_{i+1} (if the parameters are not adjacent, then U_T contains a nonempty summand $\{(\emptyset, \emptyset)\}$, and so it is not empty!). So U_T is, then $U_i = \emptyset$ for some i. Player I can never choose an element from U_i, so player II never defines a new constant that splits U_i. So the set of constants which are ever created satisfies $\sigma_{U_T} = \sigma_\emptyset = Th_{k+1}(\emptyset)$. On the other hand, if U_T is nonempty, let n be the turn immediately after the last parameter was defined. Then either there is already some constant in T, or U_T has a single summand $U_i \neq \emptyset$. Player I will play to exhaust U_i, so in particular player I will eventually play in U_i, player II will then add a constant in T.

2. Local equivalence at an element

If λ and μ are linear orders and r and s are assignments of variables or constants into λ and μ, then $(\lambda, r) \equiv (\mu, s)$ just in case the domain of r and s are the same set d, and the same logical formulas, with free variables among the elements of d, are the same in both models. This holds just in case r and s induce the same ordering on d and for each $(b, c) \in d^+$, $\{a \in \lambda : b < a < c\} \equiv \{m \in \mu : b < m < c\}$. If $\equiv$ classes of theories of linear orders respect addition, then $\equiv$ classes of formulas (of finitary logic, infinitary logic, or even non-wellfounded logic) with free variables admit addition: $\phi + \psi$ is the $\equiv$ class containing those linear orders λ and assignments s such that s assigns the free variables of ϕ and ψ into λ and for some cut $(\mu, \pi) \in \lambda^+$, s is the union of assignments t and u, where t assigns the free variables of ϕ into μ and u assigns the free variables of ψ into π, so that $\mu \models_t \phi$ and $\pi \models_u \psi$. In particular, if $\lambda \models_s \phi + \psi$, then s must assign the free variables of ϕ into λ so that they are all to the left of the free variables of ψ. We say that $\equiv$ respects addition if $\phi \equiv \phi_0$ and $\psi \equiv \psi_0$ imply that $\phi + \psi \equiv \phi_0 + \psi_0$. Finitary and infinitary quantifier-rank classes $\equiv_k$ and $\equiv_\alpha$, as well as non-wellfounded infinitary quantifier rank $\equiv_\lambda$ respect addition.

Definition 2.1. If $\equiv$ respects addition, we define left equivalence: $\phi \equiv^{\text{left}} \psi$ if there is some $\equiv$ class γ such that for all $\equiv$ classes α and β and all $\equiv$ variations $\alpha_0 \equiv \alpha$ and $\beta_0 \equiv \beta$,

$$\phi + \alpha + \gamma + \beta \equiv \psi + \alpha_0 + \gamma + \beta_0.$$

Likewise, $\phi \equiv^{\text{right}} \psi$ if there is some $\equiv$ class γ such that for all $\equiv$ and classes α and β and all $\equiv$ variations $\alpha_0 \equiv \alpha$ and $\beta_0 \equiv \beta$,

$$\alpha + \gamma + \beta + \phi \equiv \alpha_0 + \gamma + \beta_0 + \psi.$$

Finally, for any linear orders λ and μ and assignments r and s of a nonempty domain into λ and μ, we say $\lambda \equiv^{\text{loc}} \mu$ just in case

$$\lambda (\equiv^{\text{left}})^{\text{right}} \mu.$$

The following are properties of $\equiv^{\text{left}}$ and $\equiv^{\text{loc}}$:

- If $\equiv$ is an equivalence relation, then $\equiv^{\text{left}}$ is, too.
- If $\equiv$ respects addition, then $\equiv^{\text{left}}$ does, too.
- $(\equiv^{\text{left}})^{\text{left}}$ is the same as $\equiv^{\text{left}}$.
- $(\equiv^{\text{left}})^{\text{right}}$ is the same as $(\equiv^{\text{right}})^{\text{left}}$.
- If ϕ and ψ have at least one free variable, then $\phi \equiv^{\text{left}} \psi$ and $\phi \equiv^{\text{right}} \psi$ together imply $\phi \equiv \psi$.
- If ϕ and ψ have at least one free variable, then for $(\lambda, a_i)_{i<n} \in \phi$ and $(\mu, m_i)_{i<n} \in \psi$, $(\lambda, a_0, \ldots, a_{n-1}) \equiv^{\text{loc}} (\mu, m_0, \ldots, m_{n-1})$ holds just in case:
  - $\{a \in \lambda : a < a_0\} \equiv^{\text{right}} \{m \in \mu : m < m_0\}$,
  - for each $i < n-1$, $\{a \in \lambda : a_i < a < a_{i+1}\} \equiv \{m \in \mu : m_i < m < m_{i+1}\}$,
  - $\{a \in \lambda : a > a_{n-1}\} \equiv^{\text{left}} \{m \in \mu : m > m_{n-1}\}$.
- $\equiv_0^{\text{loc}}$ and $\equiv_1^{\text{loc}}$ are trivial $\equiv$ relations, i.e., for $i < 2$, $(\lambda, r) \equiv_i^{\text{loc}} (\mu, s)$ holds just in case r and s have the same domain and induce the same ordering on it.
- If $\equiv$ is an equivalence relation on theories of linear order which respects addition and if there is an $\equiv$ class σ_0 such that for every other $\equiv$ class δ it holds that $\sigma_0 + \delta + \sigma_0 \equiv \sigma_0$, then for any $\equiv^{\text{left}}$ class ϕ, the sum $\phi + \sigma_0$ is *inextensible* in the sense that for any $\equiv$ class ψ, $\phi + \sigma_0 + \psi \equiv^{\text{left}} \phi + \sigma_0$.
- If σ_0 exists as in the previous item, then for any $\equiv$ classes ϕ and ψ with no free variables, $\phi (\equiv^{\text{left}})^{\text{right}} \psi$.
- Give $\equiv_\lambda$ some linear ordering E. On a dense linear ordering η_0 such that every interval has cardinality $> |E|^{|\lambda|}$, we can define a function to E so that the sum $\sigma_0 = \sum_{a \in \eta_0} f(a)$; if $f^{-1}(a)$ is dense, then σ_0 satisfies $\sigma_0 + \delta + \sigma_0 \equiv_\lambda \sigma_0$.
- If σ_0 exists as in the previous items, then $\sigma_0 \times \omega$ or indeed any linear order γ such that $\sigma_0 + \gamma = \gamma$ is sufficient to prove $\equiv^{\text{left}}$ in the following theorem: any $\equiv^{\text{left}}$ class is $\equiv^{\text{left}} \vee U$ for U a set of inextensible $\equiv^{\text{left}}$ classes (adding any $\equiv$ class to the right leaves each of these $\equiv^{\text{left}}$ classes unchanged).
- There are three $\equiv_2^{\text{left}}$ classes of formulas with a single free variable: those (λ, a) such that a has an immediate successor, those (λ, a) such that a is the limit of a sequence descending from above, and those (λ, a) such that a is the greatest element of λ. The third $\equiv_2^{\text{left}}$ class is the disjunction of the first two, which are inextensible.

3. Labeling firsts and lasts

For any linear order λ, let $\lambda^+ = \{(b, c) : b \cup c \subseteq \lambda \text{ and } \forall d \in b \forall e \in c (d < e) \text{ and } \forall a \in \lambda ((\exists d \in b (a \leq b)) \vee (\exists d \in c (c \leq a)))\}$ modulo the equivalence $(b, c) \equiv (b', c')$ which holds just in case $\forall d \in b (\exists e \in b' (d \leq e))$ and $\forall d \in b' (\exists e \in b (d \leq e))$ be the set of *cuts* in λ. There is a natural ordering on $\lambda \cup \lambda^+$ induced from the ordering on λ: $a < (b, c)$ holds just in case $\exists d \in b (a \leq d)$; $(b, c) < (d, e)$ holds just in case $\exists f \in b (\forall g \in d (g < f))$.

If the range of a function is a linear order then that order is induced on the function itself. If b is a function with range $\subseteq \lambda \cup \lambda^+$ then let $\sup b$ be the cut $(\{a \in \lambda : \exists c \in b(a \le c)\}, \{a \in \lambda : \forall c \in b(c < a)\})$. We define the *infimum* similarly. If b and c are functions into $\lambda \cup \lambda^+$, let $\lambda^{(b,c)} = \{a \in \lambda : \sup b < a < \inf c\}$.

Definition 3.1. If I is a set of labels with any ordering and (b, c) is any cut $(b, c) \in I^+$, then for each formula τ with a single free variable (rather, for every $\equiv^{\text{loc}}$ class τ which is inextensible in the universe of $\equiv$ classes of intervals of λ) we create four new *labels*, the elements: "the least τ in (b, c)" and "the greatest τ in (b, c)" and the cuts: "the unrealized infimum of an unbounded descending sequence of τ in (b, c)" and "the unrealized supremum of an unbounded ascending sequence of τ in (b, c)." We will abbreviate these phrases with the symbols: $l\tau \in (b, c)$, $g\tau \in (b, c)$, $d\tau \in (b, c)$, and $a\tau \in (b, c)$. When the context (b, c) can be inferred, we will write them simply as $l\tau, g\tau, d\tau, a\tau$. All *labels* are constructed in a wellfounded way by this rule, from $I = \emptyset$.

For instance, if E_0 and E_1 and E_2 and E_3 are two equivalence relations on theories of linear order, and if for each $i < 4$, τ_i is an equivalence class, then $l\tau_0 \in (b, \emptyset)$, where b is a function with domain $\{l\tau_1 \in (g\tau_2 \in (\emptyset, \emptyset), g\tau_3 \in (\emptyset, \emptyset))\}$ is a label.

The element $l\tau_0 \in (b, c)$ is the least element of type τ_0 above $\sup b$ ... there is no need to refer to c, unless we want to say that the least τ_0 above $\sup b$ happens to be below $\inf c$. Viewing $l\tau_0 \in (b, c)$ as the least (or infimum of a descending sequence of) τ_0 above (say) the least τ_1 above (say) the greatest (or supremum of an ascending sequence of) τ_2 below ..., we can write the set of all labels as a tree. The first rank of the tree contains the least (or infimum) and greatest (or supremum) appearance of each type τ; the next rank of the tree contains the least appearance of each type above an element of the first rank, or the greatest appearance of each type below an element of the first rank. From the tree structure and the ordering induced by λ on its branches, we can write each label in the form given in the preceding definition.

Definition 3.2. If λ is a linear order and $\equiv$ is an equivalence relation on theories of linear order and I is an assignment of labels into $\lambda \cup \lambda^+$ then the $\equiv^{\text{loc}}$-refinement of I is the smallest assignment containing I and for each $(b, c) \in I^+$, one or two elements or cuts to indicate the definable least (or infimum) and/or greatest (or supremum) of the elements of $\equiv^{\text{loc}}$ class τ in (b, c). If τ is realized in (b, c) then τ's least element(s) are definable just in case one of the following holds:

- $b = \emptyset$, or if that fails, then
- there is a maximal element of b of any form except $a\tau' \in (e, f)$ for τ' a *high-order* equivalence class – i.e., an equivalence class in an equivalence relation which is equal to, or refines, $\equiv$, or that fails and
- elements of type τ are bounded in $a\tau' \in (e, f)$ below some element of λ.

If τ's least element(s) are definable, then I' assigns either $l\tau \in (b, c)$ or $d\tau \in (b, c)$ into $\lambda^{(b,c)}$:

- the label $l\tau \in (b, c)$ assigned to the least $h \in \lambda^{(b,c)}$ such that $(\lambda, h) \in \tau$ if there is a least such, or
- the label $d\tau \in (b, c)$ assigned to the greatest cut (g, h) such that h contains all elements of $\lambda^{(b,c)}$ of $\equiv^{\text{loc}}$ class τ, if that set h has no least element.

Similarly, for all τ realized in (b, c), the greatest element(s) of $\equiv^{\text{loc}}$ class τ are definable just in case:

- $c = \emptyset$, or $c \neq \emptyset$ and
- c has a minimal label, and that label is not $d\tau' \in (e, f)$ for τ' a high-order equivalence class, or c has either no least element or has the tricky least element just described, but
- elements of type τ are bounded in $d\tau' \in (e, f)$ above some element of λ.

If the greatest element(s) of $\equiv^{\text{loc}}$ class τ are definable, then I' contains either:

- the label $g\tau \in (b, c)$ assigned to the greatest $h \in \lambda^{(b,c)}$ such that $(\lambda, h) \in \tau$, or
- the label $a\tau \in (b, c)$ assigned to the least cut (g, h) such that g contains all elements of $\lambda^{(b,c)}$ of $\equiv^{\text{loc}}$ class τ, if that set g has no greatest element.

It is sufficient to consider only inextensible $\equiv^{\text{loc}}$ classes τ, since if $\tau = \vee U$, then if the least element of type τ is definable, then so are all the elements of U, and $\inf \tau$ is $\inf_{\tau' \in U} \inf \tau'$.

For instance, if we write e for the unique $\equiv_0^{loc}$ class, the seven $\equiv_2$ classes of linear order can be enumerated as:

$$le < ae, de < ge, de < ae, \emptyset, le = ge, le < \emptyset < ge, le < \exists x e(x) < ge.$$

That this list is a complete list of $\equiv_2$ classes of linear orders will be proved later by appeal to Theorem 0.3: a single set W contains five of the seven consistent sets of pairs of $\equiv_2$ classes, another set W contains three; the remaining nine sets of pairs of $\equiv_1$ classes are quickly proved inconsistent. For quantifier rank $k > 2$, however, Theorem 0.3 is quaint and useless; we enumerate $\equiv_k$ classes by describing trees of labels and $\equiv_{k-1}^{\text{loc}}$ classes.

Definition 3.3. Order the finite sets σ of natural numbers lexicographically: by the largest element, then the next largest, etc. Let $I_\emptyset(\lambda) = \emptyset$. For each finite set σ of natural numbers, let σ' be its immediate successor in the lexicographical order[1] and n be the least element of σ' and let $I_{\sigma'}(\lambda)$ be the $\equiv_n^{\text{loc}}$-refinement of I.

[1] The sets $\sigma < \sigma'$ are a pair of immediate predecessor and successor just in case $\sum_{i \in \sigma'} 2^i = 1 + \sum_{i \in \sigma} 2^i$.

Consider the *EF* game of length k between two linear orders μ_0 and μ_1 which realize the same $\equiv_{k-1}^{\text{loc}}$ classes. The following lemma explains how player I can use the tree of labels to find non-local differences between μ_0 and μ_1. Later, we will prove that this strategy is complete – $\equiv_k$ holds if player I cannot find a way to use this lemma; $\not\equiv_k$ holds if player I can.

Lemma 3.1. *Player I has a winning strategy in the game $EF_k(\mu_0, \mu_1)$ game if after $a_i \in \mu_i$ are chosen on the first move, the condition $(1 \wedge 2 \wedge 3) \vee (4 \wedge 5 \wedge 6)$ holds at some rank $\sigma \subseteq k-1$, $\sigma \neq k-1$, in the tree of labels*:

1. $k-2 \notin \sigma$ *and*
2. $I_\sigma(\mu_0)$ *and* $I_\sigma(\mu_1)$ *induce the same order on the same tree of labels and* a_0 *and* a_1 *are in the same cut* (b, c) *in* $(I_\sigma(\mu_i))^+$, *and*
3. *for some* $i < 2$, *some* $\equiv_{k-2}^{\text{loc}}$ *class* ρ *is realized in* μ_i *between* $\sup b$ *and* a_i *and is not realized in* μ_{1-i} *between* $\sup b$ *and* a_{1-i} *and* ρ *is definable above* $\sup b$ *in the sense of Definition 3.2, or*
4. $k-2 \in \sigma$ *and*
5. $I_{\{i:i<k-2\}}(\mu_0)$ *and* $I_{\{i:i<k-2\}}(\mu_1)$ *induce the same order on the same tree of labels and* a_0 *and* a_1 *are in the same cut* (b_0, c_0) *in* $(I_{\{i:i<k-2\}}(\mu_i))^+$, *and*
6. *for some* $\equiv_{k-2}^{\text{loc}}$ *class* ρ *and some* $i < 2$, *there is some* $p_i \in \mu_i^{(b,a_i)}$ *such that for all* $p_{1-i} \in \mu_{1-i}^{(b,a_{1-i})}$, *if* $(\mu_0, p_0) \in \rho$ *and* $(\mu_1, p_1) \in \rho$ *and* p_0 *and* p_1 *are in* (b_0, c_0) *and if* $I_{\sigma\setminus\{k-2\}}(\mu_i^{>p_i})$ *and* $I_{\sigma\setminus\{k-2\}}(\mu_i^{>p_{1-i}})$ *assign the same labels in the same order, then conditions* $(1 \wedge 2 \wedge 3) \vee (4 \wedge 5 \wedge 6)$ *hold at rank* $\sigma \setminus \{k-2\}$ *in the tree of labels after* $a_0 \in \mu_0^{>p_0}$ *and* $a_1 \in \mu_1^{>p_1}$ *are played on the first move in the game* $EF_{k-1}(\mu_0^{>p_0}, \mu_1^{>p_1})$.

Proof. Suppose condition $(1 \wedge 2 \wedge 3)$ holds. Player I plays the element of type ρ in μ_i between $\sup b$ and a_i. Player II must respond with an element in $\equiv_{k-2}^{\text{loc}}$ class ρ, since $k-2$-many moves will remain after this second move in $EF_k(\mu_0, \mu_1)$. By condition 3, player II will only find such an element below $\sup b$ in μ_{1-i}. If this were a winning second move for player II in $EF_k(\mu_0, \mu_1)$, then it is a winning first move in $EF_{k-1}(\mu_0^{<a_0}, \mu_1^{<a_1})$. But by Theorem 3.1 for $k-1$ in place of k, the first move of player II in $EF_{k-1}(\mu_0^{<a_0}, \mu_1^{<a_1})$ must be in the same interval in $I_\sigma(\mu_i^{<a_i})$; since $k-2 \notin \sigma$, $b \subseteq I_\sigma(\mu_i^{<a_i})$. Suppose, on the other hand, that conditions $(4 \wedge 5 \wedge 6)$ hold. Player I then plays the element p_i mentioned in condition 6. Player II must answer with an element p_{1-i} of the same $\equiv_{k-2}^{\text{loc}}$ class and (by Theorem 3.1 for $k=1$ in place of k) in the same interval of $I_{\{i:i<k-2\}}(\mu_i)$. If now the tree of labels $I_{\sigma\setminus\{k-2\}}(\mu_i^{>p_i})$ differ, then player II has lost; if they agree but the elements a_i are in different cuts $(b, c) \in (I_{\sigma\setminus\{k-2\}}(\mu_i^{>p_i}))^+$, then by Theorem 3.1, player II has lost. Finally, if these data are the same, then condition 6 implies that we can now re-apply the lemma to rank $\sigma \setminus \{k-2\}$ of the tree of labels $I_{\sigma\setminus\{k-2\}}(\mu_i^{>p_i})$. But since $\sigma \neq k-1$, eventually it will be conditions $1 \wedge 2 \wedge 3$ which hold, rather than conditions $4 \wedge 5 \wedge 6$. $\square$

If after the first move of the *EF* game identifies $a_i \in \mu_i$, player I finds that the lemma holds for σ and $n < k-1$ such that $n \notin \sigma$, a_i are in the same cut (b, c) of $I_{\sigma\setminus n}(\mu_i)$, and there is an $\equiv_n^{\text{loc}}$ class ρ which is definable above $\sup b$ so that an element of $\equiv_n^{\text{loc}}$ class ρ exists between $\sup b$ and a_i but not between $\sup b$ and a_{1-i} then we call ρ the *anomaly* between $\sup b$ and a_i. The following theorem's modest claim: "if player II has a winning strategy, and player II disrespects the next refinement, this leaves a game in which player I has a winning strategy" can be repeated over any series of $\equiv_n$ refinements, when n is not in the index set σ, producing an index set with n as its least element. This can produce trees with various ranks, but the one way to produce a maximal tree is to consider each $\sigma \subseteq k-1$ in lexicographical order. This then implies that if player II has a winning strategy, player II must respect $I_{\{i:i<k-1\}}$, so that $\mu_0 \equiv_k \mu_1$ implies that the same labels are sent into μ_0 and μ_1 in the same order.

Theorem 3.1. *If player II has a winning strategy in $EF_k(\mu_0, \mu_1)$, then for each $\sigma \subseteq k-1$,*

- $I_\sigma(\mu_0)$ *and* $I_\sigma(\mu_1)$ *induce the same order on the same tree of labels, and*
- *if player I plays the first move at the image of a label in one model, then either player II plays the image of that label in the other model or player I has a winning strategy in the remainder of the game, and*
- *if* $(b, c) \in (I_\sigma(\mu_0))^+$ *and player I plays the first move in* $\mu_i^{(b,c)}$, *then either player II plays in* $\mu_{1-i}^{(b,c)}$ *or player I has a winning strategy in the remainder of the game*.

Proof by induction on $\sigma \subseteq k-1$, ordered lexicographically: If $\sigma = \emptyset$, then the first two conditions require nothing and the third condition, with $(b, c) = (\emptyset, \emptyset)$ and $\mu_i^{(b,c)} = \mu_i$, is nothing more than one of the rules of the *EF* game: if player I plays in μ_i then player II answers in μ_{1-i} or loses. Now we suppose the theorem is proved up to σ_0, the immediate predecessor of σ in the lexicographical ordering, and we prove the theorem for σ itself. Since $I_{\sigma_0}(\mu_i)$ must be respected by player II on the first move, we can define a cut $(b, c) \in (I_{\sigma_0}(\mu_i))^+$ so that the first move is played between $\sup b$ and $\inf c$. To apply Lemma 3.1, note that the least element of σ is not in σ_0. The least element of σ is that n for which respecting I_σ means respecting I_{σ_0} and respecting the first and last occurrences of elements of each definable $\equiv_n^{\text{loc}}$ class. Now n is the *least* number not in σ_0 – $1 + \sum_{i<n} 2^i = 2^n$ – but the lemma applies to the *greatest* number not in σ_0. But if there is an anomalous $\equiv_n^{\text{loc}}$ class ρ between $\sup b$ and a_i, the first played element, then there is an anomalous $\equiv_{n+1}^{\text{loc}}$ class, since the $\equiv_{n+1}^{\text{loc}}$ class of any element realizing ρ is not realized in μ_{1-i} between $\sup b$ and a_{1-i}, and likewise there is an anomalous $\equiv_m^{\text{loc}}$ class, for every $m > n$.

Player I's goal is to preserve a *winning condition*: That the first moves $a_i \in \mu_i$ were played in the same cut $(b, c) \in (I_{\sigma_0}(\mu_i))^+$ and an anomaly exists – i.e., some $\equiv_m^{\text{loc}}$ class ρ is realized in μ_i between $\sup b$ and a_i and not realized in μ_{1-i} between $\sup b$ and a_{1-i}, for $n \notin \sigma_0$. Player I then plays $p_i \in I_{\{k-2\}}(\mu_i) \setminus I_{\{i:i<k-2\}}(\mu_i)$; player II must preserve $I_{\sigma_0 \setminus \{k-2\}}(\mu_i)$ above p_i, and player I finds that the anomaly has been preserved. Player I repeats this until there are $m + 1$-many moves left. That is, player I plays the first move to the lower end of the interval in $I_{\{k-2\}}$ which contains a_i, then the lower end of the interval in $I_{\{k-2,k-3\}}$ which contains a_i, and so on, until on the jth ($j = k - 1 - m$) turn player I plays the lower end of the interval (b_j, c_j) in $I_{\{k-2,k-3,\ldots,k-j\}}$ for $j \le k - 1 - m$ in which a_i occurs. We will discuss 1. how player I can play close enough to a cut and below it (or above it) to define all the $\equiv_*^{\text{loc}}$ classes which are definable above it (or below it) and 2. which model player I should play in so as to prevent new $\equiv_m^{\text{loc}}$ classes from entering the interval between $\sup b$ and a_{1-i}. We will assume, throughout, that player II plays, on the jth move, an element of the same $\equiv_j^{\text{loc}}$ class as player I. We will prove that player I can preserve a winning condition – the existence of the same anomaly – until there are $m + 1$-many moves left.

Player I's winning strategy after playing at the anomaly: On the $k - m$th move (after which there will be m-many moves remaining), player I will play the anomaly – an element of type ρ which exists between $\sup b$ and a_i, such that ρ is not realized in μ_{1-i} between $\sup b$ and a_{1-i}. Since there will remain m-many moves, player II must respond with an element of $\equiv_m^{\text{loc}}$ class ρ. This can only be found below $\sup b$ in μ_{1-i}. If player II has played the second through $k - m - 1$th moves according to a winning strategy in the *EF* game of length k, the first move of which identifies $a_i \in \mu_i$ and the $k - m - 1$th move of which identifies $p_i \in \mu_i$, then $(\mu_0^{>p_0}, a_0) \equiv_{m+1} (\mu_1^{>p_1}, a_1)$. Iterating Theorem 3.1, with k replaced by $m + 1$, for all subsets of $m + 1$, we find: $I_{\{j:j<m\}}(\mu_0^{>p_0})$ and $I_{\{j:j<m\}}(\mu_1^{>p_1})$ induce the same order on the same tree of labels and the first move, in which player I plays at the anomaly and player II plays below $\sup b \in \mu_i$, must respect $I_{\{j:j<m\}}(\mu_i^{>p_i})$. However, the labels of b which depend, in the tree of labels, on b_{k-1-m} or c_{k-1-m}, the lower and upper ends of the interval (b_{k-1-m}, c_{k-1-m}) can be redefined in terms of p_i – which was played near b_{k-1-m} with this goal in mind, or in terms of a_i, which is certainly $<c_{k-1-m}$. Thus, every label in b of tree-rank $>\sigma_0$ corresponds to a label of $I_{\{j:j<m\}}(\mu_i^{(p_i,a_i)})$, where the latter labels are mapped monotonically into λ, so that no element of $\equiv_m^{\text{loc}}$ class ρ exists between $\sup b$ and a_{1-i}, but yet player II must respect $\sup b$. So player II loses.

Player I's algorithm for determining which linear order to play in: If on the jth move (for $j \ge 1$ and $j < k - m - 1$) player I plans to play at $\sup b_j$, where (b_j, c_j) is the interval in $I_{\{k-2,k-3,\ldots,k-j\}}$ in which a_i occurs, player I must find an element close to $\sup b_j$ (see the next paragraph) in μ_i or μ_{1-i}, choosing the correct model so as to prevent player II from "widening" the interval between $\sup b$ and a_{1-i} to allow an element of type ρ to be realized there, eliminating the anomaly, or "narrowing" the interval between $\sup b$ and a_i to remove all elements of $\equiv_m^{\text{loc}}$ class ρ there.

- Player I plays $d\tau \in (b, c)$ or $l\tau \in (b, c)$ in μ_i. Player II must respond with an element of type τ, which is $\ge$ the least element of type τ in $\mu_{1-i}^{(b,c)}$, which will preserve the defined elements of b, or will shift all defined elements monotonically to the right, and so preserve or narrow the interval between $\sup b$ and a_{1-i}.
- Player I plays $a\tau \in (b, c)$ or $g\tau \in (b, c)$ in μ_{1-i}. Player II either plays the image of that label in μ_i, or plays some other, lower, realization of τ in $\mu_{1-i}^{(b,c)}$, shifting all the defined elements (in particular, all of b) monotonically to the left in μ_i. This preserves or widens the interval between $\sup b$ and a_i and preserves the anomaly.

Players can play close enough to any cut: To play the label $l\tau \in (b, c)$ or $g\tau \in (b, c)$ player I plays the image of that label. To play near $d\tau \in (b, c)$ or $a\tau \in (b, c)$ player I plays an element of type τ above $d\tau \in (b, c)$ and below $a\tau \in (b, c)$ and near the cut – closer to the cut than any other label which is not assigned to the same cut, and closer to the cut than any upper bound or lower bound which in Definition 3.2 triggers the third condition and allows some type τ' to be definable. If player I plays x_j above the lower bound on elements of type τ' below $a\tau'' \in (b'', c'')$, then the least element of type τ' above $a\tau'' \in (b'', c'')$ is the least element of type τ' above x_j. Thus, the tree of definable labels which depend on $a\tau'' \in (b'', c'')$ for their definition and are $> a\tau'' \in (b'', c'')$ is unchanged if we replace $a\tau'' \in (b'', c'')$ by x_0.

If player II does not respect I_σ, then the winning condition is established: We examine all cases and show that player II must play above some element of each $\equiv_n^{\text{loc}}$ class τ such that player I played above some element of $\equiv_n^{\text{loc}}$ class τ. On the other hand, player II must play below some element of each $\equiv_n^{\text{loc}}$ class τ which player I played below. This will imply that player II must respect the $\equiv_n^{\text{loc}}$-refinement of I_{σ_0} or lose. Thus, if the $\equiv_n^{\text{loc}}$-refinement of $I_{\sigma_0}(\mu_0)$ and the $\equiv_n^{\text{loc}}$-refinement of $I_{\sigma_0}(\mu_1)$ are not identical, player II will have lost. In particular, we now prove that in each interval of $I_{\sigma_0}(\mu_i)$, the same $\equiv_m^{\text{loc}}$ classes begin and end in the same order for $i = 0, 1$: First we prove that the same $\equiv_m^{\text{loc}}$ classes τ are realized in any cut (b, c) in $(I_{\sigma_0}(\mu_i))^+$ and then that different $\equiv_m^{\text{loc}}$ classes τ begin and end with 1. a terminal element in both linear orders or 2. no terminal element in both linear orders, and that labels for those terminal elements (or for those suprema/infima) have the same order. For if τ is realized in (b, c) in $(I_{\sigma_0}(\mu_i))^+$ and not in $\mu_{1-i}^{(b,c)}$, then player I can play τ on the first turn, player II will play below $\sup b$ (or above $\inf c$) and player I can then exhaust $\sup b$ by playing the greatest element of $b \cap I_{\{k-2,k-3,\ldots,k-j\}}$ on the jth move. If τ is realized a least time in $\mu_{1-i}^{(b,c)}$ but no least time in $\mu_i^{(b,c)}$, then player I plays the least element of type τ in $\mu_{1-i}^{(b,c)}$. Whatever element a_i player II plays, there will be an anomalous element of type τ in $\mu_i^{(b,c)}$ below a_i. Now we show that these firsts and lasts have the same order in μ_i and μ_{1-i}.

Suppose the $\equiv_n^{\text{loc}}$-refinement of $I_{\sigma_0}(\mu_i)$ maps a label to s_i, the greatest realizations of some type, or maps a label for an unrealized supremum of an ascending sequence of realizations of that type to the cut $s_i \in \mu_i$. Suppose the $\equiv_n^{\text{loc}}$-refinement of $I_{\sigma_0}(\mu_i)$ maps a label for the least realization of some type, or for an unrealized infimum of a descending sequence of realizations of that type to the cuts $r_0 \in (\mu_0)^+$ and $r_1 \in (\mu_1)^+$. We now list cases to show that the following occur in both models simultaneously: $r < s,\ r > s,\ r = s$.

- If $r < s$ holds in μ_i but not in μ_{1-i}, then player I plays the first move at a_i between r and s. By assumption, player II must respect $I_{\sigma_0}(\mu_i)$ and reply with a move $a_{1-i} \in \mu_{1-i}^{(b,c)}$ in the same cut $(b, c) \in (I_{\sigma_0}(\mu_i))^+$ for which $a_i \in \mu_i^{(b,c)}$. The type that r labels is realized in (b, c) to the left of a_i and the type that s labels is realized in (b, c) to the right of a_i. Respecting both of these conditions means playing the first move between the cuts r and s in $\mu_{1-i}^{(b,c)}$. Of course, this is only possible if those cuts have the same ordering, $r_{1-i} < s_{1-i}$.
- Suppose $s < r$ holds in μ_{1-i} and $s \geq r$ holds in μ_i. Player I plays a_{1-i} between s and r in μ_{1-i}. Whatever element a_i player II plays in $\mu_i^{(b,c)}$, either the type that r labels is realized in (b, c) to the left of a_i or the type that s labels is realized in (b, c) to the right of a_i, which is a winning condition for player I.
- Suppose $r = s$ holds in μ_{1-i} and fails in μ_i. Then $r < s$ or $r > s$ holds in μ_i. Proceed as in the previous two cases.

There remains one more case to check: suppose r and s are both the suprema of ascending sequences of different type in (b, c) in μ_i. Then $r < s$ holds just in case player I can play an element of the type which s labels, and bound all elements of type r to one side; $r = s$ holds just in case $r \not< s$ and $s \not< r$, since then they are equivalent cuts.

This theorem proves that player II must respect $I_{(\sigma_0 \cup \{n\} \setminus n)}$ if player II must respect I_{σ_0} and $n \notin \sigma_0$ and $n < k - 1$. This explains why $\equiv_k$ is equivalent to local equivalence in the intervals defined by exactly $2^{k-1} - 1$-many iterations of the refinement process. For example, if $\equiv_5$ requires player II to respect $I_{\{1,0\}}$ then we can prove that player II further respects the $\equiv_n^{\text{loc}}$-refinement of this for $n = 2$ or $n = 3$. If we take the $\equiv_3^{\text{loc}}$-refinement, and then the $\equiv_0^{\text{loc}}$, $\equiv_1^{\text{loc}}$, $\equiv_0^{\text{loc}}$, $\equiv_2^{\text{loc}}$, $\equiv_0^{\text{loc}}$,$\equiv_1^{\text{loc}}$, and $\equiv_0^{\text{loc}}$-refinements, we will have a tree of labels that player II must respect. But, in fact, a much larger tree of labels must be respected, and we can build it by taking the $\equiv_2^{\text{loc}}$-refinement first (and then the $\equiv_0^{\text{loc}}$, $\equiv_1^{\text{loc}}$, and $\equiv_0^{\text{loc}}$-refinements) before taking the $\equiv_3^{\text{loc}}$-refinement. It is interesting to see that the range of the smaller tree of labels is contained strictly in the range of the larger tree of labels, often with very different labels for the same element. The largest possible tree of labels which adds a singleton $\{n\}$ and removes the set n whenever $n \notin \sigma$, for σ indexing a rank in the tree, is clearly the tree given in Definition 3.3. However, when we argue that a certain label cannot be realized, we do not need the largest possible tree of labels, but only enough labels to define the one we want to discuss. For instance, we can describe the greatest limit point below x without calling it “the greatest limit point below the immediate predecessor of x."

Lemma 3.2. *For any linear order λ and finite set of natural numbers $\sigma \subseteq k - 1$, let $I_\sigma^{\text{left}}(\lambda)$ be the intersection of $I_\sigma(\mu)$ as μ ranges over the $\equiv_k^{\text{left}}$ class of λ and let $I_\sigma^{\text{right}}(\lambda)$ be the intersection of $I_\sigma(\mu)$ as μ ranges over the $\equiv_k^{\text{right}}$ class of λ. Then*

$$I_\sigma(\lambda) = I_\sigma^{\text{left}}(\lambda) \cup I_\sigma^{\text{right}}(\lambda).$$

Proof. Every label in the tree of labels depends ultimately on a rank-$\{0\}$ label which is one of $l\tau > \emptyset$ or $d\tau > \emptyset$ where $\emptyset$ refers to the left end, or $g\tau < \emptyset$ or $a\tau < \emptyset$, where $\emptyset$ refers to the right end. The first two are in $I_\sigma^{\text{left}}(\lambda)$ and the latter two are in $I_\sigma^{\text{right}}(\lambda)$. Similarly, all labels which depend on labels which depend ultimately on the left end (and in the next higher rank, on $l\tau > \emptyset$ or $d\tau > \emptyset$) are in $I_\sigma^{\text{left}}(\lambda)$ and all labels which depend on labels which depend ultimately on the right end are in $I_\sigma^{\text{right}}(\lambda)$. Of course, adding $\equiv_k$ classes to the right can extend the $\equiv_n^{\text{loc}}$ classes of the labels in $I_\sigma^{\text{left}}(\lambda)$ and thereby superficially change the definition of the label, but it cannot change where each of these labels is assigned in λ. □

Theorem 3.2. *For any linear order λ and any element $a \in \lambda$, from*

- *the ordered set A of $I_{\{i:i<k-1\}}(\lambda)$, the functions $(Th_{k-1}^{\text{loc}}(\lambda, a) : a \in A \cap \lambda)$ and $(\{Th_{k-1}^{\text{loc}}(\lambda, a) : b < a < c\} : (b, c) \in A^+)$, and*
- *the location of a in A, and the $\equiv_{k-1}^{\text{loc}}$ class of (λ, a),*

we can construct the structures in the first item, with $k - 1$ replacing k, and either $\{d \in \lambda : d < a\}$ or $\{d \in \lambda : d > a\}$ replacing λ.

Proof. The left labels $I_{\{i:i<k-2\}}^{\text{left}}(\{d \in \lambda : d < a\})$, those labels which depend for their definition on the left end of the interval $\{d \in \lambda : d < a\}$, correspond to labels of relatively extended $\equiv_n^{\text{loc}}$ classes in $I_{\{i:i<k-1\}}^{\text{left}}(\lambda)$. We can compute the $\equiv_{k-2}^{\text{loc}}$ class of each of those left labels as the truncation of Th_{k-2} of the $\equiv_{k-1}^{\text{loc}}$ class given in the first item, above, truncated at a. That truncation can be performed by altering $I_{\{i:i<k-1\}}^{\text{left}}(\lambda)$ through every possible $I_{\{i:i<k-1\}}^{\text{left}}(\lambda')$ which agrees with $I_{\{i:i<k-1\}}^{\text{left}}(\lambda)$ to the left of a, and taking $\vee$ of the $\equiv_{k-2}^{\text{loc}}$ class in each model, or by treating the $\equiv_{k-2}^{\text{loc}}$ class as a formula, and removing that part of it which is satisfied to the right of a. In this way, we replace the $\equiv_{k-2}^{\text{loc}}$ class (λ, a) with the $\equiv_{k-2}^{\text{loc}}$ class of $(\{d \in \lambda : d < a\}, b)$. The right labels $I_{\{i:i<k-2\}}^{\text{right}}(\{d \in \lambda : d < a\})$ can be read from the $\equiv_{k-1}^{\text{loc}}$ class of (λ, a). It remains to determine an ordering of the union $I = I_{\{i:i<k-2\}}^{\text{left}}(\{d \in \lambda : d < a\})$ and $I_{\{i:i<k-2\}}^{\text{right}}(\{d \in \lambda : d < a\})$, and to determine which $\equiv_{k-2}^{\text{loc}}$ classes exist in each cut in I^+.

For any cut $(b_0, b_1) \in (I_{\{i:i<k-2\}}^{\text{left}}(\{d \in \lambda : d < a\}))^+$ and for any cut $(c_0, c_1) \in (I_{\{i:i<k-2\}}^{\text{right}}(\{d \in \lambda : d < a\}))^+$, we compute from (c_0, c_1) the set C_1 of sequences $(\tau_\sigma : \sigma \subseteq k - 2)$ for which there exist elements of type τ_σ in order between (c_0, c_1) and a and the set C_0 of sequences which do not exist between (c_0, c_1) and a. We reverse these: There is an element of $\equiv^{\text{loc}}_{k-2}$ class τ below C_1 below a, but not below C_0 below a just in case for each sequence $(\tau_\sigma : \sigma \subseteq k-2)$ of $\equiv^{\text{loc}}_n$ classes, where n is the least element of $k-2$, we form the following label and find it to be $<a$:

- the least element(s) of $\equiv^{\text{loc}}_{k-2}$ class τ above b_0,
- for each $\equiv^{\text{loc}}_n$ class, in descending lexicographical order, from $\tau_{\{j:j<k-1\}}$ to $\tau_\emptyset$, the least element of type τ above the previous label.

Similarly, we invert the dependency of $\exists\tau \in (c_0 \in \ldots g\tau_0 \in (\emptyset, a), c_1 \ldots a\tau_0 \in (\emptyset, a))$ on a into a dependency of a on sequences in $I^{\text{left}}_{\{i:i<k-2\}}(\{d \in \lambda : d < a\})$. For instance, if a is very far from the left end, then $I^{\text{left}}_{\{i:i<k-2\}}(\{d \in \lambda : d < a\}) < I^{\text{right}}_{\{i:i<k-2\}}(\{d \in \lambda : d < a\})$; the total order on their union is that the one linear order simply precedes the other.

On the other hand, for each $\equiv^{\text{loc}}_{k-2}$ class τ there is a sequence of labels inverting $c \in c_0 \cup c_1$ such that there is an element of type τ above b_0 and below b_1 and above c in $\{d \in \lambda : d < a\}$ just in case the location of a in $I_{\{n:n<k-1\}}$ shows $a <$ the label. Let c' be the label on which the assignment of c depends: c is the least τ_0 above c' or the greatest τ_0 below c'. If c is in c_0, then we define the next label to be the greatest τ_0 below the previous label in the sequence. If c is in c_1, then we define the least τ_0 above the previous label. Repeat this for each label in the sequence of dependency, until c is the greatest τ_i below a (i.e., there are no more labels on which c depends). Define the next label to be the element of type τ_i closest to the previous label (where "closest" means the greatest τ_i below the label, if $c <$ the label, and where "closest" means the least τ_i above the label, if c is above the label). If $a <$ some label, then the $l\tau \in (b_0, b_1)$ mentioned in the definition of the label is an element of type τ between b_0 and b_1. If c is in c_1, then the label mentions the least element of type τ_0 above this element of type τ, and by induction, $a >$ the label implies that c' and the rest of what defines the right labels will be found between this element of type τ and a. If c is in c_0, then the label mentions the greatest element of type τ_0 below this element of type τ, and by induction, $a >$ the label implies that c' and the rest of what defines the right labels will be found between this element of type τ and a. This ends the proof of the lemma.

Now by induction on subsets σ of $k-2$, we can locate each element of $I^{\text{right}}_{\{i:i<k-2\}}(\{d \in \lambda : d < a\})$ within the ordering $I^{\text{left}}_{\{i:i<k-2\}}(\{d \in \lambda : d < a\})$ and determine the set of $\equiv^{\text{loc}}_{k-2}$ classes between those left and right assignments of labels. By the induction hypothesis, we know the $\equiv^{\text{loc}}_{k-2}$ classes realized between any elements of $I^{\text{left}}_{\{i:i<k-2\}}(\{d \in \lambda : d < a\})$ and of $I^{\text{right}}_{\sigma_0}(\{d \in \lambda : d < a\})$, for σ_0 the lexicographical predecessor of σ. So, a fortiori, we know the $\equiv^{\text{loc}}_{k-2}$ classes realized, for $n \le k-2$, since $\equiv^{\text{loc}}_{k-2}$ refines $\equiv^{\text{loc}}_m$. This allows us to compare, in any interval (b, c), the least τ and the greatest element of some other $\equiv^{\text{loc}}_{k-2}$ class – the least τ precedes the greatest τ_1 just in case something of type τ is realized between b and the greatest τ_1. Further, when a right label lies between left labels b and c, we can determine which $\equiv^{\text{loc}}_{k-2}$ classes are realized between $\sup b$ and the right label and which $\equiv^{\text{loc}}_{k-2}$ classes are realized between the right label and c. This ends the proof of Theorem 3.2. □

Now we prove Theorem 0.2: That $\equiv_k$ implies identical trees of labels, identical orderings A on them, and identical $\equiv^{\text{loc}}_{k-1}$ sequences on A and A^+ follows from Theorem 3.1. That $\equiv_k$ holds when these data are identical follows from Theorem 3.2, since if the data are identical, then player II can choose to play so as to respect $I_{\{i:i<k-1\}}(\lambda_i)$ and $\equiv^{\text{loc}}_{k-1}$. Then, by Theorem 3.2, data sufficient to prove $\equiv_{k-1}$ will be identical on either side of the played elements $a_i \in \lambda_i$, since this data depends only on the location of a_i within the $I_{\{i:i<k-1\}}(\lambda_i)$ and on the $\equiv^{\text{loc}}_{k-1}$ class of (λ, a_i).

4. Effective decision procedures and completions

Searching naïvely for a witness W as in Theorem 0.3 is not effective at deciding the $\equiv_3$ or $\equiv_4$ classes of linear orders, since there are $6E14$ sets of pairs of $\equiv_2$ classes and $4E23769$ sets of pairs of $\equiv_3$ classes; the number of potential witnesses is the power of those sets. We would do better to search for the data of Theorem 0.2. Then we will need an alternate game, a *local consistency game* to determine which datasets are consistent.

If U is a set of $\equiv^{\text{loc}}_{k-1}$ classes, with or without an additional single $\equiv^{\text{loc}}_{k-1}$ class on the left, and with or without an additional single $\equiv^{\text{loc}}_{k-1}$ class on the right, and V is likewise, then we define $U + V$ just in case the same $\equiv^{\text{loc}}_{k-1}$ class τ is on the right of U and on the left of V, or there is no $\equiv^{\text{loc}}_{k-1}$ class on the right of U or on the left of V. If that holds, then let $U + V = U \cup V \cup \{\tau\}$ or $U + V = U \cup V$ if there is no $\equiv^{\text{loc}}_{k-1}$ class on the right of U or on the left of V, and $U + V$ has on the left the $\equiv^{\text{loc}}_{k-1}$ class that U has on the left (if any), and $U + V$ has on the right the $\equiv^{\text{loc}}_{k-1}$ class that V has on the right (if any).

Definition 4.1. The *local consistency game* is the following: On each turn, the game state is a finite sequence $(c_i : i \le n)$ of constants, a sequence $(m_i : i \le n)$ of markers that either mark c_i as a cut or mark c_i with an $\equiv^{\text{loc}}_{k-1}$ class, and a sequence $(U_i : i < n)$ of sets of $\equiv^{\text{loc}}_{k-1}$ classes. The first player has two types of moves:

- Player I chooses $i < n$ and an element m of U_i. The second player then adds a new constant c left of c_i and right of c_{i-1}, with mark m an $\equiv^{\text{loc}}_{k-1}$ class, and chooses two sets V_0 and V_1 such that $V_0 + V_1 = U_i$ such that V_0 has the element m on the right and V_1 has the element m on the left. The game state is then the finite sequence of $n+1$-many elements and $n+2$-many sets obtained by marking c with m and replacing U_i by the pair V_0, V_1.

- Player I chooses $i \leq n$ and a label in $I^{\mathrm{loc}}_{\{i:i<k-1\}}(m_c)$, say, to the right of c_i, such that every label on which this label depends has already been played. Player II chooses a constant $c > c_i$ (c may exist already or not; c may by necessity be beyond all c_i, especially if k is large and $(U_i : i < n)$ is small – either in that it is a short sequence or in that its elements U_i are small). The constant c is marked with an $\equiv^{\mathrm{loc}}_{k-1}$ class just in case the label describes the "least" or "greatest" τ, and not the unrealized infimum or supremum of an infinite sequence of elements of type τ – player II intends that in the resulting linear order, c will be where that label, relative to c_i, must be assigned. Otherwise, the constant c is marked as a cut. Player II chooses new sets V_0, V_1 such that if V_0 is the set between c_i and the new constant c, $\{Th^{\mathrm{loc}}_{k-2}(\rho) : \rho \in V_0\}$ is the set of $\equiv^{\mathrm{loc}}_{k-2}$ classes which m_c says exists below the label.

The conditions on whether player II has survived grow as the game progresses. They are a set of conditions of two types: 1. that between certain constants c and d, all U_i ever created must omit $\equiv^{\mathrm{loc}}_{k-1}$ class τ or 2. that for a certain constant c, every interval immediately to the right of c must realize $\equiv^{\mathrm{loc}}_{k-1}$ class τ. A new condition is created every time player I plays a move of the second type. If the label is $l\tau_0 \in (b, c)$ or $d\tau_0 \in (b, c)$, then the omission condition is that τ_0 is never realized between b and the new constant. If the label is $g\tau_0 \in (b, c)$ or $a\tau_0 \in (b, c)$, then the omission condition is that τ_0 is never realized between the new constant and c. If the label is $a\tau_0 \in (b, c)$, then we require that every set ever created immediately below the new constant realizes τ_0. If the label is $d\tau_0 \in (b, c)$ then we require that every set ever created immediately above the new constant realizes τ_0. Player II survives this move if all the conditions developed so far are met.

The initial state has any number of constants with any markings, and sets U_i. For instance, the initial state could be the data of Theorem 0.2, which determine a general $\equiv_k$ class. Or the initial state could be a single set U_0 with an element on the left or not, and an element on the right or not. If the initial state has a single set U_0, then there is an initial constant c_0 on the left; it is marked with $\equiv^{\mathrm{loc}}_{k-1}$ class τ just in case U_0 has τ on the left. There is an initial constant c_1 on the right and the $\equiv^{\mathrm{loc}}_{k-1}$ class U_0 has on the right is m_1 (nothing, or an $\equiv^{\mathrm{loc}}_{k-1}$ class).

We say that player I plays an *exhaustive* strategy if player I mentions every $\equiv^{\mathrm{loc}}_{k-1}$ class in every interval U_i ever created, and mentions every label in $I^{\mathrm{loc}}_{\{j:j<k-1\}}(m_i)$ for every constant c_i ever created which is marked with an $\equiv^{\mathrm{loc}}_{k-1}$ class and not marked as a cut. If player II has a winning strategy in the consistency game and if player I plays an exhaustive strategy, then the set of constants which are not marked as cuts order grows into a model of the local classes in the initial state – a linear order λ with an element for each initial constant c_i for $i \leq n$ which is not marked as a cut, and which realizes exactly the local types in U_i for each $i < n$. On the one hand, every constant c_i marked with an $\equiv^{\mathrm{loc}}_{k-1}$ class eventually realizes that $\equiv^{\mathrm{loc}}_{k-1}$ class, as player I exhausts the labels in $I^{\mathrm{loc}}_{\{j:j<k-1\}}(m_i)$ and marks them accordingly and player II cannot violate m_i without losing. On the other hand, every element τ in every set U_i ever created corresponds to some constant c, because player I has exhausted each set U_i.

The consistency of a set U_0 of $\equiv^{\mathrm{loc}}_{k-1}$ classes can be decided quickly, since the conditions on consistency are that for each $\tau_i \in U_0$ which is postulated to exist, the labels in $I^{\mathrm{loc}}_{\{j:j<k-1\}}(\tau_i)$ have, in turn, fuller descriptions as $\equiv^{\mathrm{loc}}_{k-1}$ classes. The set of things realized between a constant c_i and its neighbors is a set $\subseteq U_0$ which omits some type, and hence is a strict subset of U_0. In this way, we reduce the question of whether U_0 is consistent to a myriad of questions about whether smaller sets are consistent. Once those smaller questions are solved, call $H_{(\tau_0,U,\mathrm{label})}$ the set of formulas $\phi_{(\rho,V)}$ which expresses that the set V of local types is realized, and ρ is realized at the end, for each pair (ρ, V) such that $\rho \in U_0$ could extend a particular label in $I^{\mathrm{loc}}_{\{j:j<k-1\}}(m_i)$ and V could be the set of $\equiv^{\mathrm{loc}}_{k-1}$ classes realized between c_i and its label ρ. U_0 is consistent just in case the Horn theory

$$\bigwedge_{\tau \in U_0} \ \bigwedge_{(\mathrm{label} \in I^{\mathrm{loc}}_{\{j:j<k-1\}}(\tau_0))} (\tau(x) \rightarrow (\bigvee_{H_{(\tau_0,U,\mathrm{label})}} \phi_{(\rho,V)}))$$

is consistent. Satisfying that Horn theory means constructing strings of $\equiv^{\mathrm{loc}}_{k-1}$ classes until the local labels of each $\equiv^{\mathrm{loc}}_{k-1}$ class is satisfied. However, as in the linear consistency game of Theorem 0.3, we can stop satisfying local labels as soon as the satisfaction process becomes repetitive. The resulting finite structures which show how to string local classes together in a model λ are the following.

Definition 4.2. An almost locally closed set is any nonempty $A \subseteq \lambda \cup \lambda^+$ such that for each $a \in A$ there is some $a_0 \in A$ such that $(\lambda, a) \equiv_{k-1} (\lambda, a_0)$ and there is a homomorphism h from the ordered set $I^{\mathrm{loc}}_{\{i:i<k-1\}}(\lambda, a_0)$ into A sending $l\tau \in (b, c)$ to the least element between $h(b)$ and $h(c)$ of $\equiv^{\mathrm{loc}}_{k-1}$ class τ, and sending $d\tau \in (b, c)$ to the greatest cut (e, f) in λ^+ such that f contains every element between $\sup b$ and $\inf c$ of $\equiv^{\mathrm{loc}}_{k-1}$ class τ, and likewise for $g\tau \in (b, c)$ and $a\tau \in (b, c)$. For each label $d\tau \in (b, c)$ or $a\tau \in (b, c)$ of $I^{\mathrm{loc}}_{\{i:i<k-1\}}(\lambda, a_0)$, A also contains an "example": an element of type τ above $d\tau \in (b, c)$ (or an element of type τ below $a\tau \in (b, c)$) such that for any $g \in \lambda$ between the cut and the example, there is an $h \in A$ not between the example and the cut, such that $(\lambda, c, d, g) \equiv_{k-1} (\lambda, c, d, h)$. For each $\equiv^{\mathrm{loc}}_{k-2}$ class τ which $Th^{\mathrm{loc}}_{k-1}(\lambda, a_0)$ knows to exist between two labels, A contains an *example*: an element of type τ between h of those two labels.

Among the almost locally closed sets which contain an element with $\equiv_{k-1}^{\mathrm{loc}}$ class τ, we are especially interested in the minimal sets, i.e., those sets A for which there is no almost locally closed proper subset of A which also realizes τ.

These sets help us to define a finitely axiomatizable linear order in any $\equiv_k$ class. We call σ a *cut-state* if it contains a single set U_0 of $\equiv_{k-1}^{\mathrm{loc}}$ classes, with or without an $\equiv_{k-1}^{\mathrm{loc}}$ class on the left end and with or without an $\equiv_{k-1}^{\mathrm{loc}}$ class on the right end, since it is the state of the linear consistency game at a cut $(\{c_{i-1}\},\{c_i\})$ in the set of constants. We call a cut-state σ *consistent* if player II can win the linear consistency game in which $\sigma=(c_0,c_1),(m_0,m_1),(U_0)$ is the initial condition. For each consistent cut-state σ we will write a sentence δ_σ which expresses that the elements of σ are realized densely. We might be tempted to insist that for all $x_0\in\lambda$ and for all $x_1\in\lambda$ (if (λ,x_0,x_1) satisfies σ, then $(\lambda,x_0,x_1)\models\delta_\sigma^{(x_0,x_1)}$). This invites a study of the consistency of a family $\{\delta_\sigma:\sigma\in W\}$ of complete sentences, on overlapping intervals. If the consistency of such a set of sentences can be decided, then we can define completions without $\equiv^{\mathrm{loc}}$ classes. But Theorem 0.2 allows us to construct a generic $\equiv_k$ class of linear order in stages, as in the linear consistency game, while controlling only the local $\equiv_{k-1}^{\mathrm{loc}}$ classes in each gap, and not the set of pairs of $\equiv_k$ classes realized in each gap. Admittedly, the theory of linear order can be described, decided, and completed, using overlapping intervals as the basic building block. Instead, we use local neighborhoods as the basic building block with which we construct, decide, and complete theories of linear order.

Definition 4.3. Suppose $x_0\in\lambda$, $x_1\in\lambda$ and there is an almost locally closed set $A\subseteq\lambda$ containing $\{x_0,x_1\}\subseteq A$ such that (λ,x_0,x_1) satisfies σ, and A is minimal among the almost locally closed sets containing $\{x_0,x_1\}$. Then let δ_σ be

$$(\exists x_a : a\in A, x_0<a<x_1)$$

$$\bigwedge_{a,b\in A\cap\lambda,\text{adjacent}} \delta^{(x_a,x_b)}_{Th^{\mathrm{loc}}_{k-1}(\lambda,x_0),\{Th^{\mathrm{loc}}_{k-1}(\lambda,x):x_0<x<x_1\},Th^{\mathrm{loc}}_{k-1}(\lambda,x_1)}.$$

If σ indicates a nonempty set of $\equiv_{k-1}^{\mathrm{loc}}$ classes between x_0 and x_1, then for τ the $\equiv_m^{\mathrm{loc}}$ class of any element of σ, the label $l\tau>x_0$ or $d\tau>x_0$ will be assigned below x_1. On the other hand, each element of A is defined in relation to some other element of A, so that in each gap between elements of A, some part of σ is not realized. Therefore we have defined δ_σ in terms of $\{\delta_\rho:\rho$ is a proper subset of $\sigma\}$.

Definition 4.4. Suppose σ is not satisfied within one minimal almost locally closed class, as was the case in the preceding definition. Suppose $x_0\in\lambda$ and $y_1\in\lambda^+$ (and consider the possibilities, too, that the left end is a cut or the right end is an element) and that (λ,x_0,y_1) satisfies the cut-state σ. Since $x_0\in\lambda$, find an almost locally closed set $A_0\subseteq\lambda$ containing x_0. Let δ_σ assert the existence of A_0:

$$(\exists x_a : a\in A_0\cap\lambda, x_0<a<y_1)($$

$$\bigwedge_{a,b\in A_0\cap\lambda,\text{adjacent}} \delta^{(x_a,x_b)}_{Th^{\mathrm{loc}}_{k-1}(\lambda,x_a),\{Th^{\mathrm{loc}}_{k-1}(\lambda,x):x_a<x<x_b\},Th^{\mathrm{loc}}_{k-1}(\lambda,x_b)}).$$

Let σ_{-A_0} be a set of almost locally closed sets (subsets of λ) such that σ requires the existence of exactly the $\equiv_{k-1}^{\mathrm{loc}}$ classes $\cup\{\{Th^{\mathrm{loc}}_{k-1}(\lambda,a):a\in A\}:A\in\sigma_{-A_0}\}\cup\{Th^{\mathrm{loc}}_{k-1}(\lambda,a):a\in A_0,x_0<a\}$. Let δ_σ assert that every x in (x_0,y_1) is $\equiv_{k-1}$ to a first move made in A_0, or is part of an almost locally closed set $\equiv_{k-1}$ to something in σ_{-A_0} :

$$\forall x((x_0<x<y_1)\to(\exists z((\vee_{b\in A_0}z<x_b)\wedge(\lambda,x_0,x,y_1)\equiv_{k-1}(\lambda,x_0,z,y_1)))\vee$$

$$(\vee_{A\in\sigma_{-A_0}}(\exists x_a : a\in A\cap\lambda)(\vee_{a\in A\cap\lambda}x_a=x)\wedge$$

$$\bigwedge_{a,b\in A\cap\lambda,\text{adjacent}} \delta^{(x_a,x_b)}_{Th^{\mathrm{loc}}_{k-1}(\lambda,x_a),\{Th^{\mathrm{loc}}_{k-1}(\lambda,x):x_a<x<x_b\},Th^{\mathrm{loc}}_{k-1}(\lambda,x_b)})).$$

Let δ_σ further assert that above A_0 the elements of σ_{-A_0} are realized without either upper or lower bound, so that every element is $\equiv_{k-1}$ to a first move in A_0 or all possible elements of σ_{-A_0} are realized below it, and that, without condition (since $y_0\in\lambda^+$), all elements of σ_{-A_0} are realized above it:

$$\forall x((x_0<x<y_1)\to($$

$$((\exists z((\vee_{b\in A_0}z<x_b)\wedge(\lambda,x_0,x,y_1)\equiv_{k-1}(\lambda,x_0,z,y_1)))\vee$$

$$(\wedge_{A\in\sigma_{-A_0}}((\exists x_a : a\in A\cap\lambda)(x_0<x_a<x)\wedge$$

$$\bigwedge_{a,b\in A\cap\lambda,\text{adjacent}} \delta^{(x_a,x_b)}_{Th^{\mathrm{loc}}_{k-1}(\lambda,x_a),\{Th^{\mathrm{loc}}_{k-1}(\lambda,x):x_a<x<x_b\},Th^{\mathrm{loc}}_{k-1}(\lambda,x_b)})))$$

$$\wedge(\wedge_{A\in\sigma_{-A_0}}((\exists x_a : a \in A\cap\lambda)(x < x_a < y_1)\wedge$$

$$\bigwedge_{a,b\in A\cap\lambda,\text{adjacent}} \delta^{(x_a,x_b)}_{Th^{\text{loc}}_{k-1}(\lambda,x_a),\{Th^{\text{loc}}_{k-1}(\lambda,x):x_a<x<x_b\},Th^{\text{loc}}_{k-1}(\lambda,x_b)})))).$$

Let δ_σ further require that the elements of σ_{-A_0} are realized densely – for every pair of elements $x < y$, if x is not $\equiv_{k-1}$ to a first move played in A_0 and such that $\{x, y\}$ is not spanned by a single element of σ_{-A_0} then every element of σ_{-A_0} is realized between x and y:

$$\forall x\forall y((x_0 < x < y < y_1) \to ($$

$$(\exists z((\vee_{b\in A_0} z < x_b) \wedge ((\lambda, x_0, x, y_1) \equiv_{k-1} (\lambda, x_0, z, y_1))))$$

$$\vee \vee_{A\in\sigma_{-A_0}} ((\exists x_a : a \in A\cap\lambda)$$

$$((\bigwedge_{a,b\in A\cap\lambda,\text{adjacent}} \delta^{(x_a,x_b)}_{Th^{\text{loc}}_{k-1}(\lambda,x_a),\{Th^{\text{loc}}_{k-1}(\lambda,x):x_a<x<x_b\},Th^{\text{loc}}_{k-1}(\lambda,x_b)})$$

$$\wedge(\vee_{a\in A\cap\lambda} x_a = x) \wedge (\vee_{a\in A\cap\lambda} x_a = y)))$$

$$\vee(\wedge_{A\in\sigma_{-A_0}}((\exists x_a : a \in A\cap\lambda)((x < x_a < y)\wedge$$

$$\bigwedge_{a,b\in A\cap\lambda,\text{adjacent}} \delta^{(x_a,x_b)}_{Th^{\text{loc}}_{k-1}(\lambda,x_a),\{Th^{\text{loc}}_{k-1}(\lambda,x):x_a<x<x_b\},Th^{\text{loc}}_{k-1}(\lambda,x_b)}))))).$$

The conjunction of the foregoing four sentences is what we call δ_σ.

If σ indicates that no $\equiv^{\text{loc}}_{k-1}$ classes are realized between x_0 and x_1, then $\delta_\sigma = Th_k(\emptyset)\wedge$ some information about the local class of the left end (if there is one) to the right of the interval, and information about the local class of the right end (if there is one) to the left of the interval. For example, if there is an $\equiv^{\text{loc}}_{k-1}$ class on the left and an $\equiv^{\text{loc}}_{k-1}$ class on the right, but σ is empty, then these two classes can be realized at a pair of immediate predecessor and successor. The formula δ_σ is then much like an $\equiv^{\text{loc}}_{k-1}$ class, in that it determines an $\equiv^{\text{left}}_{k-1}$ class right of the pair, and an $\equiv^{\text{right}}_{k-1}$ class left of the pair. If σ has no $\equiv^{\text{loc}}_{k-1}$ class on the left or on the right and is empty, then δ_σ is $Th_k(\emptyset)$ since the second sentence in the definition above says that every x is part of an almost locally closed set in σ_{-A_0}: $\forall x((y_0 < x < y_1) \to \vee\emptyset)$ is $Th(\emptyset)$.

Now we define, by induction again on σ, a model of δ_σ. Suppose that models of δ_ρ exist whenever ρ is a proper subset of σ.

Definition 4.5. If $A \subseteq \lambda$ is an almost locally closed set, minimal among those realizing a particular $\equiv^{\text{loc}}_{k-1}$ class, then let μ_A contain $A\cap\lambda$ and a copy of λ_ρ in every cut in A^+ (note that we include all of A, not only $A\cap\lambda$, on purpose) in which the set ρ is realized, perhaps with an $\equiv^{\text{loc}}_{k-1}$ class on the left or an $\equiv^{\text{loc}}_{k-1}$ class on the right. Let the least and greatest elements of A be a_0 and a_1. By the definition of an almost locally closed set, there exist $b_0, b_1 \in A$ such that $(\lambda, a_i) \equiv_{k-1} (\lambda, b_i)$. Let the half-open interval in μ_A between a_i and b_i, including a_i and not b_i, be μ_i. Let $\lambda_A = \mu_0 \times \omega^* + \mu_A + \mu_1 \times \omega$. In this way we make out of an almost locally closed set an interval for the linear order λ_A.

Suppose σ is as in Definition 4.3, i.e., that there is a single almost locally closed set A such that if σ describes an $\equiv^{\text{loc}}_{k-1}$ class on the left, there is some $x_0 \in A$ such that (λ_A, x_0) is in that class, and if σ describes an $\equiv^{\text{loc}}_{k-1}$ class on the right, there is some $x_1 \in A$ such that (λ_A, x_1) is in that class, and such that between x_0 (or $x_0 = (\emptyset, \lambda_A)$, if σ does not describe an $\equiv^{\text{loc}}_{k-1}$ class on the left) and x_1 (or $x_1 = (\lambda_A, \emptyset)$, if σ does not describe an $\equiv^{\text{loc}}_{k-1}$ class on the right), (λ_A, x_0, x_1) satisfies σ, and such that A is minimal among all almost locally closed sets containing two elements x_0, x_1 of the given $\equiv^{\text{loc}}_{k-1}$ classes. Then choose that $A \subseteq \lambda$ and elements x_0, x_1 for which 4.3 defines the density formula δ_σ and add to λ_σ constants for elements of A realizing the $\equiv^{\text{loc}}_{k-1}$ class that σ describes on the left, if there is one, and the $\equiv^{\text{loc}}_{k-1}$ class that σ describes on the right, if there is one. For instance, if σ describes neither a left nor a right element, then $\lambda_\delta = \lambda_A$.

Suppose otherwise, i.e., that no one almost locally closed set A, minimal among those that realize two $\equiv^{\text{loc}}_{k-1}$ classes, realizes every $\equiv^{\text{loc}}_{k-1}$ class described by δ. If σ has the $\equiv^{\text{loc}}_{k-1}$ class $|tau$ on the left and no $\equiv^{\text{loc}}_{k-1}$ class on the right, let A_0 be the minimal almost locally closed set containing an element $a_0 \in A_0$ of type σ as chosen in Definition 4.4, let σ_{-A_0} be the set of almost locally closed sets chosen in Definition 4.4, and let λ_δ be $(\lambda_A, a_0)+$ a dense shuffle of $\{\lambda_A : A \in \sigma_{-A_0}\}$.

Theorem 4.1. *If σ is a set of $\equiv^{\text{loc}}_{k-1}$ classes with or without a single $\equiv^{\text{loc}}_{k-1}$ class on the left and with or without a single $\equiv^{\text{loc}}_{k-1}$ class on the right, λ_σ is a linear order with one ore more constants which satisfies δ_σ, with the constants satisfying δ_σ's single type on the left or right; the remaining $\equiv^{\text{loc}}_{k-1}$ classes are realized between these constants.*

Proof. If σ is satisfied within a single linear order λ_A, for A a single almost locally closed set, then λ_σ was chosen in the previous definition's first paragraph to satisfy σ. If σ is not satisfied within a single λ_A, then δ_σ requires that the $\equiv^{\mathrm{loc}}_{k-1}$ class on the left is part of a minimal almost locally closed set A_0 – the second paragraph of the previous definition defines λ_{A_0} – and δ_σ requires that the δ_σ requires something similar on the right. Finally, δ_σ requires that almost locally closed sets in σ_{-A_0} be densely ordered. In the last paragraph of the previous definition, we find that λ_δ does in fact densely order $\{\lambda_A : A \in \sigma_{-A_0}\}$. It remains to check that in each interval in each λ_A, between any a_0 and $a_1 \in A$, such that the set ρ of $\equiv^{\mathrm{loc}}_{k-1}$ classes is realized between a_0 and a_1, then λ_ρ, with constants for a_0 and a_1 if they are in λ and not λ^+, satisfies δ_ρ with the constants a_i interpreting whatever $\equiv^{\mathrm{loc}}_{k-1}$ classes ρ requires on the left and right. By induction on strict subsets of σ, we may assume this is true. As the base case, if σ is empty, then δ_ρ describes the empty set and λ_ρ is the empty set. $\square$

By Theorem 0.2, $\mu_0 \equiv_k \mu_1$ holds just in case μ_0 and μ_1 have certain data in common. That data form an initial state of the local consistency game. So, for any $\equiv_k$ set, we take this initial state, extend its $\equiv^{\mathrm{loc}}_{k-1}$ classes to almost locally closed sets, and write δ_σ. For any finite number k, for any linear order λ, the $\equiv_k$ class of λ is determined by one such initial state for the local consistency game. For such a state, we form λ_σ in each interval, for $\sigma = U_i$, and by cutting the models λ_σ at the constants which refer to the locations of the $\equiv^{\mathrm{loc}}_{k-1}$ classes on the left of U_i and the right of U_{i-1}, we form a model of the entire initial state. A subsequence $(c_i : i_0 < i < i_1)$ of the constants may be close, in that the sets U_i between them is small. Then it is likely that a single almost locally closed set which is minimal among almost locally closed sets containing even c_{i_0} will contain them all. We could then form the single set λ_A to explain the whole sequence $(c_{i_0}, U_{i_0}, c_{i_0+1}, \ldots, c_{i_1})$, though the theoretically simpler definition is simply to form λ_A on each triple c_i, U_i, c_{i+1}, to cut it at the constants for c_i and c_{i+1}, and to add these linear orders together. In any case, wherever there are constants $c_i < c_{i+1}$ such that no one almost locally closed set which is minimal among those containing elements with the $\equiv^{\mathrm{loc}}_{k-1}$ classes of c_i and c_{i+1} realizes all of U_i, then we choose an almost locally closed set A_i which is minimal among those containing an element with the $\equiv^{\mathrm{loc}}_{k-1}$ class of c_i and an almost locally closed set A_{i+1} which is minimal among those containing an element with the $\equiv^{\mathrm{loc}}_{k-1}$ class of c_{i+1} and realize the $\equiv^{\mathrm{loc}}_{k-1}$ classes left over from U_i densely between (λ_{A_i}, c_i) and $(\lambda_{A_{i+1}}, c_{i+1})$. By the following theorem, we are justified in calling the linear order which is the piecewise sum, over constants $(c_i, i \leq n)$, of dense linear orders λ_δ, the *piecewise-dense* model $PWD(Th_k(\lambda))$ of $Th_k(\lambda)$.

Theorem 4.2. *Let k be any finite number; let λ be any linear order. Then the following equivalence holds: $\lambda \equiv_k PWD(Th_k(\lambda))$ and $PWD(Th_k(\lambda))$ is finitely axiomatized by*

$$Th_k(\lambda) \wedge \bigwedge_{a,b \in I_{\{i:i<k-1\}}(\lambda),\text{adjacent}} \delta^{(a,b)}_{\{Th^{\mathrm{loc}}_{k-1}(\lambda,x):a<x<b\}},$$

where instead of postulating the existence of elements x_a for each a in the set of indices $I_{\{i:i<k-1\}}(\lambda)$ and restricting δ to occur between x_a and x_b, we instead use the fact that a and b are definable elements and cuts in λ, and we restrict δ to occur among the set of elements $d \in \lambda$ such that d is above the defined cut a and below the defined cut b.

Proof. As in the previous theorem, $PWD(Th_k(\lambda))$ satisfies δ_σ for each adjacent pair of elements $a, b \in I_{\{i:i<k-1\}}(\lambda)$, where δ describes the $\equiv^{\mathrm{loc}}_{k-1}$ class classes realized between the defined elements or cuts a and b. That is, Th_k of the interval between a and b only requires that the local classes in σ be realized between the definable elements or cuts a and b. The formula $\delta^{(a,b)}_\sigma$ adds to this a choice about how the local classes form into almost locally closed sets, insists that this happens regularly (uniformly within the interval (a, b)), and insists that these almost locally closed sets are realized densely. But since this certainly implies that exactly the $\equiv^{\mathrm{loc}}_{k-1}$ classes in δ are realized, for any first move played in (a, b) in λ or $PWD(Th_k(\lambda))$, player II can answer with an element which has its $\equiv^{\mathrm{loc}}_{k-1}$ class in δ and which is realized between a and b. This implies by Theorem 3.2 that the linear orders left and right of the played elements are $\equiv_{k-1}$. To see that $\lambda \equiv_k PWD(Th_k(\lambda))$ is finitely axiomatized by the given formula, we will compute its $\equiv_{k+m}$ class from the given formula, for any natural number m. By Theorem 0.2, the $\equiv_{k+m}$ class is determined by sequences $(Th^{\mathrm{loc}}_{k+m-1}(\lambda, a) : a \in I_{\{i:i<k+m-1\}}(\lambda) \cap \lambda)$ and $(\{Th^{\mathrm{loc}}_{k+m-1}(\lambda, a) : b < a < c\} : (b, c) \in (I_{\{i:i<k+m-1\}}(\lambda))^+)$. For each natural number m, $I_{\{k+m\}}(\lambda)$ adds labels for the first and last occurrence of each $\equiv^{\mathrm{loc}}_{k+m}$ class. Those $\equiv^{\mathrm{loc}}_{k+m}$ classes which are realized in λ_{A_i} for A_i the almost locally closed element containing the ith element of $I_{\{i:i<k-1\}}(\lambda)$, are realized as λ_A orders them – by induction on proper subsets $\rho \subseteq \sigma$, the intervals in λ_A determine where these $\equiv^{\mathrm{loc}}_{k+m}$ classes begin and end. The remaining $\equiv^{\mathrm{loc}}_{k+m}$ classes which are realized in $PWD(Th_k(\lambda))$ are realized in λ_A for $A \in \sigma_{-A_0}$ (or $A \in \sigma_{-A_0,-A_1}$), sets of almost locally closed sets which realize all $\equiv^{\mathrm{loc}}_{k-1}$ classes not realized in A_0 (or not realized in A_0 or in A_1). Again, by induction on proper subsets of σ, we can determine which $\equiv^{\mathrm{loc}}_{k+m}$ classes are realized in λ_A. It is easy to see that density will force these $\equiv^{\mathrm{loc}}_{k+m}$ classes to be realized without lower or upper bound and to be realized all the way down to the cut $(\lambda_{A_0}$, the shuffle of $\sigma_{-A_0})$. $\square$

5. Semimodels

Unlike previous sections, this section describes a topic without the motivation of a main theorem to which everything trends. Instead we gather together some results on *semimodels*, a rich concept.

Theorem 0.3 shows that for any consistent set U of pairs of $\equiv_k$ classes, a finite set W of information witnesses the consistency of U. The proof of Theorem 0.3 adds this twist: if player I plays an exhaustive strategy in the linear consistency game in Definition 1.1, and if player II plays according to a function (f_0, f_1) from W to $W \times W$, then other variations in player I's strategy have no effect on the linear order λ which is created during the game – each element of λ is waiting to be created because W defines it in terms of other elements of λ which are waiting to be created, and the order in which player I goes about turning these into played constants does not change the set of elements which are ultimately created, nor its ordering. That is, "λ is built in stages according to $W, (f_0, f_1)$" is a complete description of λ. However, we went ahead and defined $PWD(Th_k(\lambda))$ and proved in Theorem 4.2 that this λ has a complete description in first-order logic over the vocabulary $<$. The simpler definition "λ is built in stages according to $W, (f_0, f_1)$" can be expressed over the vocabulary $<$ in a logic which is first-order and has the additional capacity that it recognizes "stages."

Given any linear order λ, for each pair x_0, x_1 of elements of λ, let $U(\lambda, x_0, x_1) = U(Th_{k+1}(\{x_2 \in \lambda : x_0 < x_2 < x_1\}))$ be the set of pairs of $\equiv_k$ classes realized as $(Th_k(x_0, x_2), Th_k(x_2, x_1))$ for various x_2 between x_0 and x_1. The relationship between $W, (f_0, f_1)$ and the $\equiv_{k+1}$ class $U(\lambda)$ of λ is complicated. If it were possible to interpret each complete description $W, (f_0, f_1)$ as a set $U(\lambda)$, we would have a complete theory in the given $\equiv_{k+1}$ class. Theorem 0.3 checks that $Th_{k+1}(\lambda)$ is consistent by finding that for any constants x_0 and x_1 which are defined and adjacent at some stage, those constants define an interval (x_0, x_1) (similarly, one constant which was at some stage the least defined constant defines intervals with no left endpoint or no right endpoint, and no constants define the entire linear order itself) such that $U(x_0, x_1)$ is an element of a set $W = \{U_I : I \text{ is an interval in } \lambda\}$ so that for $U \in W$ and for any of U's elements, e.g., $(Th_k(x_0, x_2), Th_k(x_2, x_1)) \in U_{(x_0, x_1)}$, there exists a pair (V_0, V_1) of elements of W such that $(\xi(V_0), \xi(V_1)) = (Th_k(x_0, x_2), Th_k(x_2, x_1))$ and $V_0 + (\emptyset, \emptyset) + V_1 = U_{(x_0, x_1)}$. In λ, there must exist an element x_2 between x_0 and x_1 such that $Th_k(x_0, x_2) = \xi(V_0)$ and $Th_k(x_2, x_1) = \xi(V_1)$. There must exist Skolem functions f_0 and f_1 for the formula $\forall U \in W(\forall(\phi, \psi) \in U(\exists V_0 \exists V_1(((\xi(V_0), \xi(V_1)) = (\phi, \psi)) \wedge (V_0 + (\emptyset, \emptyset) + V_1 = U_{(x_0, x_1)}))))$, which expresses consistency. But for triples $x_0, x_1, x_2 \in \lambda$ such that x_0 and x_1 were never adjacent during the construction of λ, x_0 and x_1 can have the same set $U_{(x_0, x_1)}$ and the triples x_0, x_1, x_2 can have the same $(Th_k(x_0, x_2), Th_k(x_2, x_1))$, while the pair $(U(x_0, x_2), U(x_2, x_1))$ may be very different from the value

$$(f_0(U_{(x_0, x_1)}, (Th_k(x_0, x_2), Th_k(x_2, x_1))), f_1(U_{(x_0, x_1)}, (Th_k(x_0, x_2), Th_k(x_2, x_1))))$$

of those Skolem functions! If λ has been created in stages according to the functions (f_0, f_1) from W to $W \times W$, then λ has some triples which obey (f_0, f_1) (at least, those triples x_0, x_1, x_2 for which x_0 and x_1 were at one point adjacent), but λ might well have triples which do not obey (f_0, f_1). That every triple in λ obeys $W, (f_0, f_1)$ can be expressed in first-order logic as $\forall x_0(\forall x_1(\chi))$ where χ is the formula:

$$\wedge_{U \in W}(\sigma_U^{(x_0, x_1)} \rightarrow (\forall x(\wedge_{(\phi, \psi) \in U}((\phi^{(x_0, x)} \wedge \psi^{(x, x2)}) \rightarrow$$

$$((f_0(U, (\phi, \psi)))^{(x_0, x)} \wedge (f_1(U, (\phi, \psi)))^{(x, x_1)})))))$$

If this formula is consistent, it finitely axiomatizes any model λ. For we could compute Th_{k+m} of any interval (x_0, x_1) in λ as σ_Q where Q is the set of pairs of Th_{k+m-1} theories of intervals (x_0, x) and (x, x_1) for various x in the interval (x_0, x_1).

That is, if player II has a winning strategy in the consistency game and plays that strategy as a function, and if that strategy turns out to hold of all triples the resulting linear order is complete. Every linear order can result from player II's play in the consistency game, so long as player II adds some "randomness" to the strategy. For some initial states U_0 there are sets W which prove that U is consistent, so there is always a function f which constructs a linear order. But in general, there are pairs x_0, x_1 which arise in the tree of constants constructed during play which were never neighbors during the game, and yet which, at the end of the game, have the same state $U_{(x_0, x_1)}$ as some pair y_0, y_1 which were neighbors during the game. How can we describe the linear order which is built during the consistency game in which player II plays a strategy which is a function? Semimodels are a good way to describe, up to $\equiv$, models which are built according to repeated rules, because they address the notion of "stages."

Definition 5.1 [3]. A semimodel is a nested sequence $(M_i : i < \omega)$ of finite sets with a common ordering on $\cup_{i<\omega} M_i$. We call $(M_i : i < k)$ the rank k part of M. We say $M \models^{\text{semi}} \phi$ if $(M_i : i < \omega) \models \phi^{\text{semi}}$, where ϕ^{semi} is the relativization of ϕ in which we replace any subformula $\exists x \psi$ of ϕ which occurs within the scope of n-many quantifiers by $\exists x((x \in M_n) \wedge \psi)$ and we replace any subformula $\forall x \psi$ of ϕ which occurs within the scope of n-many quantifiers by $\forall x((x \in M_n) \rightarrow \psi)$.

Let $\chi_1 = ((\forall x_0 \forall x_1 \chi) \wedge (\forall x_1 \forall x_0 \chi))$.

Now $\{\chi_1, \exists y_0(\chi_1), \exists y_0(\exists y_1(\chi_1)), \ldots\}$, the set containing χ_1 with any number of dummy quantifiers prepended in front of χ_1, describes up to $\cong$ the countable semimodel built by iterating the Skolem functions f_0 and f_1. Thus, every $\equiv_k$ class contains a linear order with a simple semimodel description. Semimodels were introduced with the following theorem in mind:

Theorem 5.1 [3]. *If U is a class of finite semimodels, and the following hold*:

1. *U is recursively enumerable*.
2. *For any formula ϕ and any semimodel $L \in U$ such that the rank of L as a semimodel is the rank of ϕ as a formula, if $L \models^{\text{semi}} \phi$, then L extends to a full linear order λ, such that $\lambda \models \phi$*.

3. *For any formula ϕ and any linear order λ such that $\lambda \models \phi$, ϕ^{semi} holds in some semimodel $L \in U$,*

then the theory of linear order is decidable.

Proof. Enumerate the implications of the theory of linear order, and look for $\neg\phi$. Meanwhile, enumerate elements of U, and look for $L \in U$ of rank equal to the quantifier rank of ϕ, such that $L \models \phi$. By condition 2, some linear order λ models ϕ, too. By condition 3, if ϕ is consistent, then this procedure terminates in the discovery of a semimodel of ϕ. $\square$

The second condition rejects a number of intuitive semimodels, if the semimodels $(M_i : i < \omega)$ which are defined by repetitive play of a winning strategy in a consistency game are intuitive and if $(M_i : i < \omega)$ extends to $\cup_{i<\omega} M_i$. These structures are described up to $\cong$ in the class of countable semimodels, by a theory which affixes dummy quantifiers to the formula χ_1, given above. If there were a model of any element of that semi-theory, there is a model of the whole theory, since dummy variables do not alter whether a model satisfies a sentence, or not. The model would be finitely axiomatized by χ_1. For some winning strategies in the consistency game, there is no model of χ_1. These semimodels must be excluded from U. Even if χ_1 is not consistent, the semimodels $(M_i : i < \omega)$ have a consistent union and a simple semi-theory, even though the theory of the linear order $\cup_{i<\omega} M_i$ is not χ, i.e., for some natural number k, it holds, for many winning strategies in the consistency game, that $(M_i : i < k) \not\equiv_k^{\text{semi}} -1 \cup_{i<\omega} M_i$. We write the rank k part of a semimodel as the sequence $(s_i : i < n)$ where s_i is the least number n such that the ith element of M_{k-1} is M_n. From such a sequence we recover the semimodel's stages as $M_j = \{i : s_i \leq j\}$. We add semimodels by concatenating their sequences – i.e., we write one after the other. We multiply semimodels $N \times M$ by replacing every element of M of rank i by a copy of N in which every number has been increased by i. This usually produces a lot of waste which we can then trim away, finding a smaller sequence which is $\equiv_k$.

Lemma 5.1. *If M is a semimodel and μ is a model, then the following are equivalent*:

- *For every sentence ϕ of quantifier rank k, $M \models^{\text{semi}} \phi$ just in case $\mu \models \phi$.*
- *Player II has a winning move in the EF game between M and μ, where on the jth move, any move played in M must be played in M_j.*

Proof. If there is a sentence ϕ violating the first item, then player I can use that as a winning strategy in item II. On the other hand, from a winning strategy for player I in item II we can create a formula ϕ which is satisfied in μ just in case it is not semimodel satisfied in M. $\square$

If one or both of those conditions occur, we say $M \equiv_k^{\text{semi}} \mu$. The $\equiv_2$ classes of linear orders have semimodels:

$\emptyset, 0, 00, 000, 100, 001, 101.$

Lemma 5.2. $(0)^{2^{k-1}}(1)^{2^{k-2}}(2)^{2^{k-3}} \ldots (k-1)^{2^0} \equiv_k^{\text{semi}} \omega.$

Proof. We play the *EF* game between the semimodel and ω – in the semimodel the jth move is restricted to M_j. We answer a large natural number with the last 0. To the right, this leaves the statement of the lemma for $k-1$, which holds by induction. To the left, this leaves $(0)^{2^{k-1}-1}$ which is $\equiv_k^{\text{semi}}$ to any large, finite linear order, as the reader may wish to prove by induction. $\square$

A sequence (s_i) is an $\equiv_k^{\text{semi}}$ semimodel for the integers, Z if 1. it tapers, from 0 to k, at least as slowly as in the preceding lemma and 2. it is continuous – it never ascends from j to $j+2$ or descends from $j+2$ to j, without the value j in between.

Lemma 5.3. *If $E \equiv_k^{\text{semi}} \eta$, the countable dense linear order without endpoints, then E with every element increased by one $+(0)+E$ with every element increased by one $\equiv_{k+1}^{\text{semi}} \eta$. $101 \equiv_2^{\text{semi}} \eta$; $2120212 \equiv_3^{\text{semi}} \eta$.*

Proof. Like E, η has only one type of element. If we assign the variable x to an element of that type, the k-quantifier theory of η left of x or right of x is the k-quantifier theory of η. Base case: $\emptyset \equiv_0^{\text{semi}} \eta$. $\square$

To these examples, and the properties of $\times$ and $\sum$ for linear orders, we can also add semimodel versions of the random shuffle of a number of linear orders, and create a semimodel for any element of the hierarchy M_{LL} of [7], since that hierarchy is defined by $+$, $\times\omega$, $\times\omega^*$ and shuffle. But we do not find semimodels especially convenient for defining the shuffle – the resulting semimodel is very large, and repeats many sequences so that it is easy to define but unwieldy to work with. Instead, semimodels handle $\equiv_k^{\text{loc}}$ classes gracefully, and short semimodels in each $\equiv_k$ class can be obtained from the data of Theorem 0.2. The following theorem allows us to write semimodels in each $\equiv_k$ class which are shorter than M_{LL} semimodels, and much shorter than those expressing the Skolem functions of Theorem 0.3:

Theorem 5.2. *For any linear order μ and any finite k, there is a semimodel M which is $\equiv_k^{\text{semi}} \mu$ such that $|M_0| = |I_{\{i:i<k-1\}}(\lambda) \cap \lambda| + \sum\{|\{Th_{k+m-1}^{\text{loc}}(\lambda, a) : b < a < c\}| : (b, c) \in (I_{\{i:i<k+m-1\}}(\lambda))^+\}$ the size of M_k is bounded by $|M_0|$ times an upper bound on the size of the semimodels which express the various $\equiv_{k-1}^{\text{loc}}$ classes in μ.*

Proof. We write semimodels for $\equiv^{\mathrm{loc}}_{k-1}$ classes as follows: let 0 represent the element whose $\equiv^{\mathrm{loc}}_{k-1}$ class we wish to describe, and add on either side add semimodels for $\equiv^{\mathrm{left}}_{k-1}$ and $\equiv^{\mathrm{right}}_{k-1}$ classes. An $\equiv^{\mathrm{left}}_{k-1}$ class has as its semimodel any M such that each $\equiv_{k-1}$ class ϕ in the $\equiv^{\mathrm{left}}_{k-1}$ class has a semimodel $M + N$. A model in which a set of local types exists can be obtained by simply concatenating semimodels for those local classes. By Theorem 0.2, we know that an $\equiv_k$ class is equivalent to a sequence of elements with determined $\equiv^{\mathrm{loc}}_{k-1}$ classes, and sets of $\equiv^{\mathrm{loc}}_{k-1}$ classes between them. The concatenation of a sequence of semimodels, one for each label and one for each $\equiv^{\mathrm{loc}}_{k-1}$ class supposed to exist between the labels $\models^{\mathrm{semi}}$ the desired $\equiv_k$ class. □

The inextensible $\equiv^{\mathrm{loc}}_2$ classes of a single free variable have semimodels:

$$\{12021, 101, 1201, 1021\}.$$

An $\equiv_3$ class realizing $|U|$-many of these has a semimodel with at most $3 + 3 + |U| \times 5$ elements. This upper bound is almost tight: the smallest semimodel of the theory $le < lf < le < \{12021, 101, 1201, 1021\} < ge < gf < ge$ has 14 elements, while this theorem suggests the semimodel

$$000 + 12021 + 101 + 1201 + 1021 + 000,$$

i.e., we can eliminate 8 of the lower-order elements without affecting the $\equiv^{\mathrm{semi}}_3$ class of the semimodel.

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